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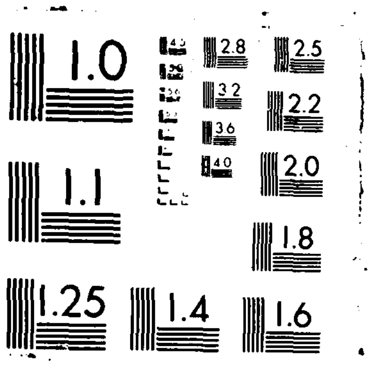
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USING SECOND-ORDER POLYNOMIALS AS PRODUCTION FUNCTIONS

THESIS

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USING SECOND-ORDER POLYNOMIALS AS PRODUCTION FUNCTIONS

THESIS

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James J. Revetta, Jr.

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1. AFIT - Air Force Institute of Technology
2. ANOVA - Analysis of Variance
3. CCD - Central Composite Design
4. CD - Cobb-Douglas (Production Function)
5. CES - Constant Elasticity of Substitution (Production Function)
6. MC - Marginal Cost
7. MSE - Mean Squared Error
8. RSM - Response Surface Methodology
9. RTS - Rate of Technical Substitution
10. SAS - Trademark, SAS Institute
11. SLAM - Simulation Language for Alternative Modeling
12. SS - Sum of Squares
13. VES - Variable Elasticity of Substitution (Production Function)

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ABSTRACT

This research attempted to show that an Air Force unit can be modeled as an industry with its "output" determined through a production function. A second-order polynomial was used as the production function in this research. A resource-allocation simulation was used to generate the data for analysis. Only two input factors were analyzed--support kits and maintenance crews. In this way, these two inputs could be compared to the microeconomic factors of production--capital and labor.

Basic Response Surface Methodology (RSM) techniques were used to estimate the second-order polynomial. Experimental designs in the form of central composite designs (CCD) were used to determine the input factor combinations. A complete statistical analysis of the pure linear model and the second-order model, complete with statistical tests and ANOVA, was performed. Basic microeconomic definitions of first- and second-order conditions were discussed and the conditions for least-cost combinations of the inputs for the second-order polynomial were derived.

A canonical analysis was done on the output data in order to plot the response contours of maximum yield. Also, a cost constraint was imposed on this production function and a multiple response surface with maximum yield and minimum cost contours overlapped within the experimental region was plotted to show the relationship of maximum yield to minimum cost.

The results of the canonical analysis of the response model indicated that a production function can be maximized subject to a minimum cost constraint through the use of a multiple response system. The path from

maximum yield to minimum cost, and the trade-offs involved, were discussed. Also, the problems associated with using simulated data to estimate production functions were outlined. Finally, some benefits of using a second-order polynomial as a production function in comparison with the commonly used Cobb-Douglas and Constant Elasticity of Substitution (CES) production functions were discussed.

USING SECOND-ORDER POLYNOMIALS

AS PRODUCTION FUNCTIONS

I. Introduction

General Issue

Often the following question is asked to Air Force decision makers:

> If your unit is given a budget increase of
> x dollars per year, where should this money
> be allocated in order to achieve the maximum
> increase in overall effectiveness for the system?

This question often arises and, in cases where the system is extremely large and complex, no immediate answer is apparent. This problem is compounded even more due to the fact that the Air Force bases many of its important cost decisions on results obtained from computer resource-allocation simulations instead of results from real-world observations. Now, more than ever, simulations are used by decision makers because: (1) In simulations, much more control can be imposed over the experimental conditions than in the real world. "What if" questions can be asked of the computer simulation model where little or no data currently exists.
(2) Much of the underlying randomness can be controlled by computer simulations by simply controlling the pseudorandom numbers that drive the stochastic events which occur in the computer simulation model. Variance-reduction techniques can be employed to yield estimators having greater

statistical precision. Thus, there is no need for complete randomness of the experimental conditions or simulation run order to guard against the introduction of biases and variation in the system. (3) Simulations may account for many detailed aspects of the actual system and therefore simulation models may easily handle exceptionally large numbers of input variables. (4) Problems caused by missing data and outliers do not cause problems in simulation studies. Since the computer simulation is a closed system, outliers can not occur. Also, with appropriate time allotment and funding, missing output data is rarely seen (37:250-251).

All of the information needed to answer the above question is contained in a simple production function from microeconomic theory. A production function expresses the relationship between the maximum quantity of output and the inputs required to produce it and also the relationship between the inputs themselves. A common production function used in industry is the Cobb-Douglas (CD) production function:

$$Q = AK^{å}L^{ß} \qquad (1)$$

where A is a positive constant and å and ß are positive fractions (11:373). Major features of this production function are: (1) It is homogeneous of degree (å+ß). (2) If (å+ß)=1, it is linear homogeneous. (3) Its isoquants are negatively sloped throughout and convex downward. (4) If (å+ß)=1, the function exhibits constant returns to scale (11:374).

Consider the function, $Q_0 = AK^{å}L^{ß}$. Taking natural logarithms of both sides:

$$\ln Q_0 = \ln(A) + å \ln(K) + ß \ln(L) \qquad (2)$$

And then the total differential:

$$dK/dL = -(dF/dL)/(dF/dk) = -(ß/L)/(å/K) = -ßK/åL \qquad (3)$$

Therefore, $-\beta K/\alpha L < 0$.

Then, the second total differential:

$$d^2K/dL^2 = d/dL(-\beta K/\alpha L) = (-\beta/\alpha)d/dL(K/L) = (-\beta/\alpha)(1/L^2)(L(dK/dL)-K) > 0.$$

The signs of these derivatives indicate the isoquants to be downward sloping and convex.

Assume, $\alpha+\beta=1$. Therefore, $\beta=1-\alpha$. Rewriting $Q_0 = AK^{\alpha}L^{1-\alpha}$ and taking second partial differentials with respect to K and L yields the marginal products:

$$dQ/dK = A\alpha K^{\alpha-1}L^{1-\alpha} = A\alpha K^{\alpha-1}L^{-(\alpha-1)} = A\alpha(K/L)^{\alpha-1} \quad (4)$$

$$dQ/dL = AK^{\alpha}(1-\alpha)L^{-\alpha} = A(1-\alpha)(K/L)^{\alpha} \quad (5)$$

Thus, economic meaning can be assigned to the values of the exponents, α and β, in the linear Cobb-Douglas production model. Since each input is to be paid by the amount of its marginal product, the relative share to capital will be:

$$(K/Q)(dQ/dK) = [KA\alpha(K/L)^{\alpha-1}]/[AK^{\alpha}L^{\beta}] = \alpha \quad (6)$$

And labor's share will be:

$$(L/Q)(dQ/dK) = [L\ A(1-\alpha)(K/L)^{\alpha}]/[AK^{\alpha}L^{\beta}] = 1-\alpha = \beta \quad (7)$$

Therefore, α and β are the relative shares of the total product of capital and labor. This indicates the need for α and β to be positive fractions and the result $\alpha+\beta = 1$ confirms exhuastion of this rule (11:374-375).

Another popular production function is the Constant Elasticity of Substitution (CES) production function. This production function is of the form that the elasticity of substitution may take on any constant value other than one. The CES function:

$$Q = A[\delta K^{-p} + (1-\delta)L^{-p}]^{-1/p} \quad (8)$$

where: $(A > 0; 0 < \delta < 1; p > -1)$.

K and L represent two factors of production and A,δ,p are three parameters (11:382). A is known as the efficiency parameter and takes on the same role as A in the Cobb-Douglas function: it declares the general state of technology. The parameter δ is the distribution parameter and, like alpha in the Cobb-Douglas function, determines relative factor shares in the product. The parameter p is the substitution parameter that determines the value of the constant elasticity of substitution (11:382).

Like the CD-function, the CES function is homogeneous of degree one and thus displays constant returns to scale. The elasticity of substitution equals (marginal function)/(average function)=(1/1+p). Therefore, the elasticity of substitution, s, is a constant whose magnitude depends on the value of the parameter p:

$$-1 < p < 0 : s > 1$$

$$p = 0 : s = 1$$

$$0 < p < \infty : s < 1.$$

Thus the Cobb-Douglas function is a special case of the CES function when p is equal to zero (11:382).

In contrast to these two production functions is a group of variable elasticity of substitution (VES) production functions in which the elasticity of substitution parameter, s, may take on a range of different values depending on the input combinations. Second-order polynomials are a common example of some VES production functions.

VES production functions have a substitution parameter which varies linearly with the capital-labor ratio around an intercept term of unity (33:64). The VES gives a linear view of the economic process in contrast to the log linear view given by the CES function (33:68). Most economic studies

assume a specified numerical value for the elasticity of substitution parameter, s. The Cobb-Douglass (CD), for instance, assumes s equals unity while most strictly linear production functions assume zero or infinity. In theory, this parameter may take on any value between zero and infinity.

The elasticity of substitution parameter can be a variable depending on the input/output combinations. Thus, an assumption of a constant elasticity of substitutions may lead to a specification bias (33:63). Revankar (1971) gave a specific CD generalization as to the choice of s:

$$s = 1 + \beta(K/L) \tag{9}$$

where β is a parameter. This equation states that s varies linearly with the capital-labor ratio. A function that exhibits this behavior is known as a VES production function. It can simply be shown that when β=0, s=1 and the VES degenerates to a CD. Therefore, the null hypothesis that β=0 is of great importance (33:64)

The biggest difference between the VES and the CES is the linear relationship among the economic variables in the VES while the CES gives log-linear relationships. The elasticity of substitution, s, for a VES function is:

$$s = s(K,L) = 1 - [(p - 1)/(1 - \partial p)][K/L] \tag{10}$$

Thus, $\beta = [(p - 1)/(1 - \partial p)]$. So, it can be shown, that s varies with the capital-labor ratio around the intercept term of unity. One must also assume that s is greater than zero in the experimental range of (K,L) (33:65).

Another important difference between the VES and the CES is the CES requires the elasticity of substitution be the same at all points of the isoquant map regardless of the level of output. The VES requires only that

the elasticity of substitution be the same along a ray in the isoquant map. The parameter may vary along an individual isoquant (33:67).

With all of this economic theory available, relating some Air Force output to a production function and an underlying cost constraint poses an interesting issue. The general issue at hand here is to treat some Air Force unit as some type of industry. Through the use of a computer simulation model, the "industry's" production function can be built. Then, finally, the production function can be optimized subject to some budget constraint.

This last point is simply the microeconomic theory of the least-cost combinations of inputs. This is what is commonly referred to as simple first-order conditions (39:1). The problem is formulated as minimizing a cost function: $C = aP_a + bP_b$ subject to some output constraint: $Q(a,b) = Q_0$. The objective fuction will then be:

$$Z = aP_a + bP_b + \mu[Q_0 - Q(a,b)] \tag{11}$$

To satisfy the first-order conditions for minimizing C, the input levels must satisfy the following simultaneous equations:

$$Z_a = P_a - \mu Q_a = 0 \tag{12}$$

$$Z_b = P_b - \mu Q_b = 0 \tag{13}$$

$$Z_\mu = Q_0 - Q(a,b) = 0 \tag{14}$$

The first two equations imply the condition:

$$P_a/Q_a = P_b/Q_b = \mu \tag{15}$$

This implies that at the least-cost combination of inputs, the input price-marginal product ratio, must be the same for each input. Therefore, the Lagrange multiplier, μ, can be said to equal the marginal cost of production.

The second-order condition readily follows (39:4). To insure minimum cost after the first-order condition is met, the production process must have a negative bordered Hessian, ie.

$$|H| = \begin{vmatrix} 0 & -Q_a & -Q_b \\ -Q_a & -\mu Q_{aa} & -\mu Q_{ab} \\ -Q_b & -\mu Q_{ab} & -\mu Q_{bb} \end{vmatrix}$$

$$= \mu(Q_{aa}Q_b^2 - 2Q_{ab}Q_aQ_b + Q_{bb}Q_a^2) < 0 \qquad (16)$$

Since the marginal cost (μ) is always positive, the expression in parentheses must be negative (39:6-13).

The first assumption that is made here is that the marginal utility is positive (Q_a and $Q_b > 0$). However, diminishing marginal utility (Q_{aa} and $Q_{bb} < 0$) alone will not ensure the expression in parentheses is negative. Each term must be noted separately.

Looking closer at the expression in parentheses, Q_{ab} will exist only if there is an interaction term (ie. x_1x_2) in the production function equation. If there is not an interaction term then the entire middle term will be zero. This will force the expression ($Q_{aa}Q_b^2 + Q_{bb}Q_a^2$) to be negative. Since the squared terms are always positive, both Q_{aa} and Q_{bb} must be negative for the second-order conditions to be met.

If an interaction term does exist, it has the possibility of being either positive or negative. This term describes how increases in x_1 affect the marginal utility of x_2. In general, it is not possible to predict the sign of this interaction term (30:93). If it is positive, th middle term is negative and if Q_{aa} and/or Q_{bb} are positive, $|2Q_{ab}Q_aQ_b|$ must be greater than $|Q_{aa}Q_b^2 + Q_{bb}Q_a^2|$ to ensure that second order conditions are satisfied. If the interaction term is negative, then Q_{aa} and Q_{bb} must be negative and

$|Q_{aa}Q_b^2 + Q_{bb}Q_a^2|$ must be greater than $|2Q_{ab}Q_aQ_b|$.

Now this process can be evaluated in terms of the elasticity of substitution parameter, s. When the input price ratio (P_a/P_b) rises, the optimal input ratio of b/a will also rise because input b (now relatively cheaper than a) will tend to be substituted for input a. The elasticity of substitution can measure the extent of this substitution (11:381-382).

$$
\begin{aligned}
s &= (\%\text{ change in } b/a)/(\%\text{ change in } P_a/P_b) \\
&= [d(b/a)/(b/a)]/[d(P_a/P_b)/(P_a/P_b)] \\
&= [d(b/a)/d(P_a/P_b)]/[(b/a)/(P_a/P_b)]
\end{aligned}
\tag{17}
$$

The value of s can be anywhere between zero and infinity. The larger the value of s, the greater the substitutability between a and b. For the Cobb-Douglas function, $(b/a)=(\beta/\alpha)(P_a/P_b)$. Thus, $d(b/a)/d(P_a/P_b)=\beta/\alpha$ and $(b/a)/(P_a/P_b)=\beta/\alpha$. Substitution into the elasticity equation yields $s=1$.

Therefore, the Cobb-Douglas equation is characterized by a constant unitary elasticity of substitution. This result does not rely on the fact that $\alpha+\beta=1$. Thus, the elasticity of substitution of the production function will be unity even if $\alpha+\beta$ does not equal 1 (11:375).

This least-cost combination can be constructed for a second-order polynomial as a production function (assuming no interaction term) in the following manner:

Assume the production function of interest is:

$$y = b_0 + b_1x_1 + b_2x_2 + b_{11}x_1^2 + b_{22}x_2^2 \tag{18}$$

A few assumptions must be made. (1) y, the output, can be treated as a response function. (2) the parameters, b_i's, can be estimated by the method of least squares. (3) conditions of cost-minimization and output-maximization can be met. Therefore,

$$f_1 = b_1 + 2b_{11}x_1 > 0 \text{ and } f_{11} = 2b_{11} \tag{19}$$

$$f_2 = b_2 + 2b_{22}x_2 > 0 \text{ and } f_{22} = 2b_{22} \tag{20}$$

The resulting second-order conditions derived in Eq (16) must be met in order for a least-cost combination of inputs to result. Eqs (19) and (20) are the result of a closed-form solution only in the absence of an interaction term, f_{12}. As mentioned earlier, if f_{12} is absent, then f_{11} and f_{22} must be negative in order for the entire expression to be negative. If f_{12} is not equal to zero, then the sign of the middle term must be taken into account. If f_{12} is positive, then the entire middle term will be negative since f_1 and f_2 must always be positive. Thus, since f_{11} and f_{22} are negative, the entire expression in Eq (16) will be negative and cost minimization will be possible.

But, if f_{12} is negative, the entire middle term will be positive, and so the relative magnitude of the middle term with respect to the two negative end terms must be calculated to ensure that cost minimization is possible. However, there is not a closed form solution, using this method, for finding the cost minimization conditions with a interaction term present in the second order polynomial production function.

The augmented objective function can be written as a cost function:

$$\text{Min } C = P_1x_1 + P_2x_2 \tag{21}$$

subject to an output constraint as a production function:

$$y = f(x_1,x_2) = b_0 + b_1x_1 + b_2x_2 + b_{11}x_1^2 + b_{22}x_2^2 \tag{22}$$

Then using the method of Lagrange multipliers,

$$Z = P_1x_1 + P_2x_2 + \mu[y-(b_0 + b_1x_1 + b_2x_2 + b_{11}x_1^2 + b_{22}x_2^2)] \tag{23}$$

where μ is the Lagrange multiplier and in this case can be used as the marginal cost.

Taking derivatives:

$$dZ/dx_1 = p_1 - \mu f_1 = 0 \text{ and } dx_2/dp_2 = p_2 - \mu f_2 = 0 \quad (24)$$

$$dZ/d\mu = -\mu(b_0 + b_1x_1 + b_2x_2 + b_{11}x_1^2 + b_{22}x_2^2) = 0 \quad (25)$$

$$\mu = p_1/f_1 = p_1/(b_1 + 2b_{11}x_1) = p_2/f_2 = p_2/(b_2 + 2b_{22}x_2) \quad (26)$$

and setting the marginal costs equal to each other:

$$p_1/(b_1 + 2b_{11}x_1) = p_2/(b_2 + 2b_{22}x_2) \quad (27)$$

Thus, the input price ratio is:

$$p_1/p_2 = (b_1 + 2b_{11}x_1)/(b_2 + 2b_{22}x_2) \quad (28)$$

where $p_1 = \mu b_1 + 2\mu b_{11}x_1$ and $p_2 = \mu b_2 + 2\mu b_{22}x_2$ and where $2\mu b_{11}x_1 = p_1 - \mu b_1$ and $2\mu b_{22}x_2 = p_2 - \mu b_2$.

So, $x_1 = (p_1/2\mu b_{11}) - (b_1/2b_{11})$ and $x_2 = (p_2/2\mu b_{22}) - (b_2/2b_{22})$ where the demand equations for maximizing the utility of the inputs are:

$$x_1^* = x_1(p_1, p_2, y; b_i\text{'s}) \quad (29)$$

$$x_2^* = x_2(p_1, p_2, y; b_i\text{'s}) \quad (30)$$

$$\mu^* = \mu(p_1, p_2, y; b_i\text{'s}) \quad (31)$$

These demand equations show that the optimal choice of an input is not only a function of its own price but also the prices of the other input, given a constant output, and the parameters of the model (30:131).

Substituting back into the original production function, Eq (19):

$$\begin{aligned} y &= b_0 + b_1x_1 + b_2x_2 + b_{11}x_1^2 + b_{22}x_2^2 \\ &= b_0 + b_1[(p_1/2\mu b_{11}) - (b_1/2b_{11})] + b_2[(p_2/2\mu b_{22}) - (b_2/2b_{22})] + \\ &\quad b_{11}[(p_1/2\mu b_{11}) - (b_1/2b_{11})]^2 + b_{22}[(p_2/2\mu b_{22}) - (b_2/2b_{22})]^2 \quad (32) \end{aligned}$$

Carrying out all the arithmetic, a positive real root is possible if and only if:

$$4b_{11}b_{22}y + b_1^2b_{22} + b_2^2b_{11} < 0 \text{ or} \quad (33)$$

$$4b_{11}b_{22}y < -[b_1^2b_{22} + b_2^2b_{11}] \text{ or} \quad (34)$$

$$y < -[(b_1^2b_{22} + b_2^2b_{11})/(4b_{11}b_{22})] \quad (35)$$

Cost minimization is possible if the negative bordered Hessian |H| is greater than zero or $(1/f_2^2)(f_{11}f_2^2 - 2f_1f_2f_{12} + f_{22}f_1^2) < 0$.

It was shown that: $f_1 = b_1 + 2b_{11}x_1 > 0$ and $f_{11} = 2b_{11}$, $f_2 = b_2 + 2b_{22}x_2 > 0$ and $f_{22} = 2b_{22}$. Using the above function, $f_{12} = 0$, so: $(2b_{11})(b_2 + 2b_{22}x_2)^2 + (2b_{22})(b_1 + 2b_{11}x_1)^2 < 0$. Since the squared terms are always positive, cost minimization will be possible if:

$$(b_1 + 2b_{11}x_1)^2/(b_2 + 2b_{22}x_2)^2 < -(b_{22}/b_{11}) \qquad (36)$$

Again, it is important to mention that then second-order polynomial production function used in this example does not contain an interaction term, x_1x_2. If such an interaction term does exist, then the aforementioned method of Lagrange multipliers can be applied in a search for the least-cost combination of inputs, however the solutions will be different.

Specific Problem Statement

In this research, a production function will be treated as a response surface and basic response surface methodology (RSM) techniques will be used to derive a second-order polynomial production function from simulated data.

This polynomial will be of great help to the analyst because it will free him of the constraints and complications encountered when working with either the Cobb-Douglas or CES production functions. The basic problem boils down to predicting future responses from an existing response (production) function.

The constraints on the Cobb-Douglas function hamper its use. Although this function is linear in its logarithmic form and simple least-squares regression can be used to estimate its parameters, the elasticity of substitution is unity for any input factor combination and for any capital

intensity (b/a). This function is therefore incapable of representing a change in the case of substitution of labor for capital (10:42).

Although the CES production function is an extension of the Cobb-Douglas function, some complications with the CES limit its use. The major problem is that it is difficult to generalize the function for more than two input factors. Also, the CES function assumes the elasticity of substitution between capital and labor does not vary with respect to the input factors. Finally, and probably most importantly, the CES function is difficult to fit to data since its parameters are nonlinear. Therefore, a nonlinear least-squares iterative computer program involving initial guesses as to the values of the parameters is required for parameter estimation (10:62).

The most common and easily interpretted form of a response function is a straight line relationship. However, most data, whether from simulation, lab experimentation, or real-life observation, do not follow straight lines but rather, curves. There are two methods for the study of a curvilinear relationship. One is a transformation of the variables (both dependent and independent) so the resulting relationhip between the variables is linear. The other is to work directly with the curvilinear relationship. The end result of the two methods is usually the same. However, it is usually easier for one to think in terms of a curved response than in terms of some type of lograithmic transformation (24:816).

Some benefits and problems associated with using polynomials as response functions can now be discussed. The most commonly used response function is a polynomial of degree p. The simple straight line is when p=1. Asides from this, the most common is the quadratic function (p=2) (24:816).

Quadratics are popular because they only involve addition of an extra term to the straight line relationship making them the simplest curvilinear relationship. Also, from high school calculus, quadratics have a simple optimum a $x = -B_1/2B_2$. The common method of least-squares can estimate the parameters of this function easily. Polynomials make easy generalizations to multidimensional relationships between a response y and several x-variables. This allows for increased flexibility because of the powers allowed in the x-variables and because the quadratic polynomial is not symmetric (24:817).

Some disadvantages associated with polynomials are that extrapolation is virtually impossible outside the range of the constrained x-values. This sometimes results in impossible values of y predicted with only a small degree of extrapolation outside the range of the x-values. Also, linear polynomials are symmetric about the optimum value. Finally, asymptotic forms of relationships cannot be formed by quadratics (24:817).

Review of the Literature

Topic.

It is the purpose of this literature search to review the published findings in the field of nonlinear production functions and response surface methodology (RSM) techniques. This review will be limited to the two most common functions--the Cobb-Douglas and the Constant Elasticity of Substitution (CES) production functions and the class of second-order polynomials as response functions.

Background.

As mentioned earlier, a production function expresses the relationship between the maximum quantity of output and the inputs required to produce it and also the relationship between the inputs themselves. In analysis, a production function can be treated as a response surface. Decision making is constrained by the technology embedded in the response surface. In the first part of this review, two commonly used nonlinear functions, the Cobb-Douglas and the CES production functions, will be analyzed as response surfaces. All production functions embody technological constraints that are imposed on the decision making process. But the decisions are not imposed on the way in which outputs relate to inputs. There are four characteristics of the production function which are extremely useful for analysis: technological efficiency, economies of scale, the degree of intensity of one input of a technology, and the ease with which one input is substituted for another (10).

Technological efficiency refers only to the relationship between inputs and outputs. For a given set of inputs, the efficiency along with other

factors, determine the resulting output. It is simply a scalar transformation of inputs into outputs.

Economies of scale are defined as: for a given proportional increase in all inputs, when output is increased by a larger proportion then the process exhibits increasing returns (or economies of scale). If output is increased by the same proportion then the process exhibits constant returns. And if output is increased by a smaller proportion then the process exhibits decreasing returns (or diseconomies of scale) (10:13).

The degree of intensity is defined as the quantity of one input relative to the quantity of another input used in the production process. The concern here is with the technological requirements of the production process, not the levels of relative input supplies.

The ease with which one input is substituted for another is defined as the elasticity of substitution. In the two functions analyzed here, the Cobb-Douglas function is characterized by an elasticity of substitution strictly equal to one while the CES function can have any constant elasticity value. Due to the strict equality of the Cobb-Douglas' elasticity of substitution, this function is incapable of representing a change in the fourth characteristic of production functions--ease of substitution. Hence, an extension to this simple nonlinear function is the CES. However, it was described earlier how the CES production function is not always the easiest function to employ.

The second part of this review will be a discussion of the literature dealing with response surface methodology (RSM). RSM theory will be the basis for the building of the production function used in this research.

Justification.

Because of the extensive use of simulations in analysis, many decisions are based on results obtained from a system simulation model. In a resource allocation simulation, an optimal mix of inputs is found to produce a desired level of output. Validation of this simulation would be an attempt to demonstrate that the simulation behaves like the actual system. The most compelling objection to simulation arises here: the difficulty of distinguishing good results from bad ones (36:101). This problem could develop into a major concern--expensive experimentation with a poor simulation and allocation of resources based on the results.

Scope.

In this literature review, the current articles on parameter estimation, elasticity of substitution, efficiency, and economies of scale are reviewed in an effort to show the complicated issues involved when dealing with nonlinear production functions. Then, through the review of the RSM literature, the advantages of using second-order polynomials as response (production) functions in the place of a Cobb-Douglas or CES production function will be apparent.

Nonlinear Production Functions.

The first step in the use of any production function in the input-output process is the estimation of its parameters. The functions that will be discussed in the first half of this literature review are nonlinear in nature. While the Cobb-Douglas function is linear in its parameters, the CES function is nonlinear in its parameters. In the last several years increasing

acceptance to the fact that nonlinear models and nonlinear estimation problems can be handled, if not quite as routinely as linear problems, at least in a reasonably effective manner has led to many interesting situations.

Bodkin and Klein (1967) used two approaches to estimating the nonlinear parameters: direct (single equation) estimates of the two principle variants of an aggregate production function and estimates obtained from a two equation system. They note that production functions and associated marginal productivity are essentially nonlinear relationships and used computer programs to cut through these nonlinearities to obtain direct estimates of the parameters. When estimating the Cobb-Douglas production function, the two procedures yield very similar results. However the CES production function comparison yields different results. Due to large negative intercorrelation between the residuals, the two-step procedure yields different results from those based upon simultaneous equations (2:38). They conclude that with either formulation of the error terms (additive or multiplicative), the CES is a strongly nonlinear function of the parameters which cannot be made linear by a logarithmic or other simple transformation. The natural procedure in this case would seem to use nonlinear methods of estimation (2:33).

Eisenpress and Greenstad (1966) developed a computer package that handles nonlinear estimation procedures by solving each equation in a nonlinear system by ordinary least squares and then to use these results as the initial approximations to the full information solution. One drawback to their procedure is if nonlinear least squares estimation procedures are applied to such an equation, and the results used in the second stage, it is assumed that the errors are additive (15:860). This is a poor assumption

since in at least one test computation, the two stage estimates turned out to be poorer than the ordinary least squares estimates (15:861).

A different approach to estimating the Cobb-Douglas function was introduced by Goldberger (1968). His main focus is that when the Cobb-Douglas form is used, the standard specification and approach to estimation shift attention from the mean to the median as a measure of central tendency (17:464). He shows how minimum variance unbiased estimators of parameters of either the mean or median may be obtained. Since taking logarithms gives a linear regression relationship with a different intercept, which measure of central tendency to be used should be explicit (17:467).

Kumar and Gapinski (1974) used the nonlinear least squares regression technique to try and handle the econometric characteristics of the CES estimators. Simulated data resulted in smooth response surfaces but these results are considered dubious due to a high degree of multicollinearity that may occur in actual practice (20:563). When actual data was run several different variances of the estimates were found. The variances could be so large as to suggest a stochastic series or small enough to be virtually deterministic. Their results show little bias in the parameter estimates, except for the elasticity of substitution, regardless of whether the true response surface has additive or multiplicative errors (20:564). The elasticity of substitution, on the other hand, was estimated very imprecisely (20:564). One cause of this imprecision was how the regression program operates near the optimum. However, the study is very useful in showing that nonlinear least squares estimation is an important tool in estimating the parameters of the CES production function. Nonlinear least squares appears

to be an important tool for providing accurate estimates except for the elasticity of substitution (20:567).

The driving force in most of the literature concerning nonlinear estimation of the CES function is Kmenta (1967). He found using a simultaneous equation technique that when constant returns to scale is specified the elasticity of substitution can be estimated from the marginal productivity condition by regressing the value of production per one input on the parameter estimate of another (both measured in logarithms) (19:180). However, if the CES function is generalized to permit nonconstant returns to scale, the method is no longer feasible. He found that when data is available exhibiting nonconstant returns to scale, the CES function is clearly preferable over the restrictive Cobb-Douglas function (19:186).

One subject that is of much debate is the value that the elasticity of substitution takes on in a simulated production function.

In his article, Thursby (1980) compares a new parameter estimation technique to three known procedures. Adopting the Kmenta approximation methods he found reliable estimators for each of the parameters except the elasticity of substitution. The difficulty is that the expected value of the estimator does not exist under certain conditions and, when it does, the variance may be extremely large (38:295). Kumar and Gapinski (1974) reported that while most of the parameters were estimated with small bias and variance, estimates of the elasticity of substitution were completely unreliable (38:296).

Corbo (1977) states the problem is one of assumptions of either profit maximization and constant returns to scale or cost minimization. Thus, it is impossible to know if the parameter that one is estimating is the elasticity of

substitution or some other parameter resulting from untested assumptions (12:1466). Using Kmentas' approach, he found reliable estimates of the returns to scale parameter but not reliable estimates of the elasticity of substitution (12:1466). He concludes that when one wished to choose among different production models, it is not proper to use the Kmenta approximation to test whether the production function is indeed a CES (12:1467).

Berndt (1976) also commented on the "substantial disagreement over the value of the elasticity of substitution (1:69)." He found the discrepency occurring over differences in data. Studies based on cross-sectional data gave estimates close to unity while time series studies reported lower estimates (1:59). Regression based on the marginal product of one input produced lower estimates than regressions based on the marginal product of another input. Berndt concludes that estimates of the elasticity of substitution are extremely sensitive to differences in measurement and data construction (1:59).

Maddala and Kadane (1967) hypothesized that a misspecification of the elasticity of substitution could result in biases in the estimates of returns to scale. The question was asked: "Suppose that the true production function is the CES function with constant returns to scale but elasticity of substitution significantly different from unity. We estimate, however, the Cobb-Douglas production function instead. Do we observe increasing, constant, or decreasing returns to scale (22:420)?" Their major conclusion is that misspecification of the elasticity of substitution can result in a substantial bias in the estimates of returns to scale (22:420). Again, Kmenta's procedure was used to estimate both parameters simultaneously. This procedure was

not found to give reliable estimates of the elasticity of substitution although reliable estimates of the returns to scale parameter were found. Their study concludes with saying that if one is studying economic growth, then returns to scale is the important parameter and the Kmenta approximation to the CES function (even if it does not give a reliable estimate of the elasticity of substitution) is preferable to using the Cobb-Douglas production function (22:420).

In trying to come to grasp with misspecifications involving the estimation of production elasticities, Meeusen and van den Broeck (1977) specified a model for a production function based on errors due to inefficiency, statistical errors due to randomness, and to specification and measurement errors. They state that the aforementioned least-squares estimation techniques systematically underestimate the values for production efficiency (25:436). The reason for this is that the least-squares estimation ignores statistical error. Their parameter estimates and constant returns to scale are consistent with previous Cobb-Douglas estimates however no sensitivity analysis was done to ensure this efficiency parameter holds over a wide range of the response surface (25:443).

Response Surface Methodology.

The beginning of the study of response surface methodology, commonly referred to as RSM, began in the 1951 with a critical paper written by G.E.P Box and K.B. Wilson. Box and Wilson (1951) used experimental designs to find a point on a response surface which maximizes output or yield. The advantage to their method was that the experimental design chosen uses the smallest number of observations possible. They compare full and fractional factorial designs on the basis of precision and bias and introduce the concept of central composite designs (CCD) for the first time. This paper layed the foundation for the sequential movement, through experimental designs, from a first order model to a second- or higher-order model. A CCD has the advantage that it estimates all the derivatives up to the second order with equal precision (7:16). The method of steepest ascent to search for a near-stationary region around the optimum point of the response surface was formed (7:23). The dominant assumption in this paper was that responses can be estimated by a polynomial by simply varying the levels of the input variables in the experimental design. These different experimental designs are compared in terms of the variance-covariance matrices of their parameter estimates (7:25).

The Box and Wilson paper spawned a series of related and expanded works throughout the 1950's. These papers were the building blocks of the entire RSM technique that is used today. Box and Hunter (1957) introduced the concept of rotatability for a response surface design. Rotatability refers the variance function of the response surface being spherical. This is important since the variance of a predicted response will depend only on the distance of the prediction point from the center of the design and not on its direction (6:204).

Combining this thought with the initial concept of the CCD, Box and Behnken (1960) showed that the minimum-varianced estimator of the model is that with a spherical variance function--a rotatable design (8:456). They also showed that for a design to be rotatable, it must be orthogonal. They then proved that each factor in the design must be varied at only three levels to approximate a polynomial response surface (8:459). Before this paper, most thought that to approximate a polynomial response function, each factor had to be varied at five levels. Box and Behnken's paper resulted in reducing the amount of experimental runs needed to approximate the response (8:460-463). The first to show this was DeBaun (1959). He showed completely rotatable central composite designs requiring each factor be varied at five levels and then proved it was possible to construct a second-order design where only three levels of each factor are required (13:4). Although the variance of a design with five x-levels will always be less than a design with three x-levels, DeBaun showed that a three-level design can be an efficient enough estimate of the response surface. This is advantageous since, in some experimental designs, it would be very difficult to vary the factors at five levels. Three levels (-1,0,+1) make it much more "convenient" for the experimenter and still produce sufficient response surfaces.

Most of the early work in RSM was focused mainly on factorial designs to maximize a response in a given region of interest. The first to do work in the area of describing the shape of the response surface were Box and Draper (1959). Their work covered designs from exploring response surfaces to specific designs for estimating parameters and then a methodology to distinguish between the two types of response surface models (4:624).

Mead (1975) asked two common questions: (1) How does one choose which model gives the best fit? and (2) If one has fit some model to

different sets of data how does the investigation proceed to prove which models are different (24:821)? Draper and Smith (1966) showed that both of these questions can be answered with linear models because their corresponding sampling distributions are known. In most situations, model comparison becomes difficult since the term "best-fit" is ill-defined and the analysis required to provide a comparison is not available (24:821). Draper and Smith first defined the technique for linear models: (1) The inclusion or exclusion of terms can be tested using extra sum of squares. (2) For models with equal numbers of parameters, a simple comparison of the residual sum of squares can be made. (3) When the purpose of fitting a model is to predict future responses, Mallows (1973) developed the C_p criterion:

$$C_p = (\text{Residual SS}/s^2)-(N-2p) \tag{37}$$

to choose the variables to be included in the final model (24:821). This is a modification of the extra sum of squares technique but allows for the number of parameters to be fitted (23:664). These three techniques will be used in this research when the final model determination is to be made.

An important point in RSM that is often overlooked is the source of error in the experimental design. Box and Wilson (1951) were the first to describe the two sources of error in experimental designs--experimental error and bias error. They showed that when an experimental design is being considered, both variance of the predicted response function and the difference between the two response functions should be considered (7:34). A method for the choice of the design was first described by Box and Hunter (1957). They suggested that the estimated function should approximate the true function as closely as possible within the experimental region. Also, the design should be a check for the accuracy of the estimated response function. The design should not contain a large number of experimental points.

Finally, it must be possible to easily extend the response function to a higher order with minimal effort (6:197). Box and Draper (1959) then extended this work to show that the optimal design is where both the variance and bias error is the same. The experimenter can then ignore the variance error and simply design the experiment to minimize bias (4:473).

Often, experimental error is not sufficiently large enough to require the precision that is supplied by a complete factorial design. Also, higher order interactions can often times be ignored. Fractional factorial designs can be used in these situations. The main use of fractional factorials is to reduce the dimension of the experiment by examining the marginal response surfaces independent of certain effects. Box and Wilson (1951) showed that if the experimenter has some prior knowledge concerning the shape of the response surface in the experimental region then only one experiment is needed at each design point. But, if there is little knowledge, a cruder model may be used to simply get a rough picture of the response surface. This may be important when the experimenter is more interested in how the response behaves upon movement away from the optimum conditions than in an actual response function to approximate the experimental region. Thus it is the shape of the response surface and not an exact polynomial from the central composite design that may be of interest to the experimenter (7:309).

To aid in this dilemma, Box (1954) introduced the process of canonical analysis and showed the method of interpretting the signs of the coefficients in the new coordinate system (3:35). The main reason for a canonical analysis would be to gain information on the nature of the response surface. Also, it can be easily and quickly determined if a true maximum, true minimum, or a saddle point results at the stationary point in the

experimental region (3:36). Canonical analysis will be a main feature in the analysis done for this research.

Research Objectives

It will be the purpose of this research to model a typical Air Force unit as an industry and build its production function from computer simulated data. Experimental design and response surface techniques will be used to build a second-order polynomial as the response surface to be used as the production function.

This research will then show the convenient use of RSM techniques to show if the "industry" is maximizing its output within some cost constraint. Response surface contours, obtained through a canonical analysis of the system, will be used for demonstrating this output maximization-cost minimization process only if all of the criteria for cost minimization are satisfied.

Finally, this research will show both the advantages and disadvantages of using second-order polynomials as production functions. Also, the benefits of using these variable elasticity of substitution (VES) functions will be discussed in relation to the Cobb-Douglas and CES production functions.

Overview

The remainder of this thesis consists of three chapters. Chapter II gives a verbal description of the methodology, both statistical and micro-economical, used in developing the multiple response surfaces to be analyzed.

Chapter III analyzes the findings of the response surface least-squares fit. In addition, the steps involved in developing the canonical analysis and

the output maximization and the cost-minimization response contours are discussed.

The final chapter, Chapter IV, presents the contour plots for the maximum output and the minimum cost response surfaces and also the multiple response contour plot. Conclusions reached during this research and recommendations for further research are also discussed.

II. Thesis Methodology

This research effort will combine the statistical methods of least-squares regression, experimental design, and response surface methodology with the microeconomic properties of output maximization and cost minimization in the form of production functions. Both of these will be outlined in the following methodology.

Response Surface Methodology, usually referred to as RSM, is a statistical and mathematical method in which a system product is influenced by a number of input variables. Now the system product will be termed the response variable. This is also commonly known as the dependent variable in a mathematical equation. All of the input variables can be termed the independent variables. The response variable is usually some type of measure of cost or yield and the the way in which the inputs vary will determine the magnitude of this response (27:445).

In this research effort, a typical Air Force simulation will be analyzed. Since most Air Force decisions are based on results obtained through a simulation model, analysts often stake their reputation on these results. The typical scenario discussed here will be a simple input-output simulation in which a response will be affected by the combinations of the different input variables. Although most analysts will simply try to optimize the response with its associated inputs a second question can be asked. Is this combination of inputs the least-cost combination that will maximize the response? In other words, is the Air Force maximizing its "production" while also minimizing its costs? The word production is used in the sense that the Air Force can be treated as a production firm or industry. The Air Force produces an output and while this product is not sold in the competitive

market, the Air Force must be concerned, as are all non-profit organizations, with producing its product at minimum cost.

This research will be an attempt to use RSM techniques to build an Air Force production function and then through these same techniques devise a method of representing the maximum yield and minimum costs combination of inputs simultaneously. This method will be a valuable tool for analysts who are tasked to decide on the optimal input mix that will maximize a response with cost considerations taken into account. Also, the benefits and hazards of using a second-degree polynomial as a production function will be discussed.

Background

Before one can begin a discussion of response surface techniques, a review of least squares regression and experimental design is in order. Both of these techniques are used heavily in any RSM research.

The basic problem is that a response, y, is a function of n independent variables. The actual form of this response is unknown but for the purpose of this research it will be assumed to be approximated by a low order (second degree) polynomial (28:61). For the case of two independent variables, the response could be approximated by the model:

$$y = b_0 + b_1x_1 + b_2x_2 + b_{11}x_1^2 + b_{22}x_2^2 + b_{12}x_1x_2 + e \tag{38}$$

where the b_i's are the parameter estimates, y is the response variable, and e is the random error in the model. The x_i's should be both quantitative and continuous. It will be assumed that the errors are independent with zero mean and constant variance (28:62). As a useful notation the linear model will be written in matrix form:

$$y = XB + e \tag{39}$$

The least squares procedure will be useful in estimating the parameters in the vector B. This is commonly referred to as linear regression. The researcher will know the values of the X matrix since it is a function of the selected x levels and the y vector will be a column of responses. The responses, y_i's, can be obtained from any experimental procedure. In this research effort, the y_i's are obtained from repeated computer simulation runs at certain combinations of the x_i's set up in a convenient experimental design which will be discussed later. Then the least squares method for estimating B will be one which minimizes the sum of squares of the errors or the deviations the estimated values take from the observed responses. A minimum value for L can be obtained from:

$$L = \Sigma e_i^2 = e'e \tag{40}$$

Now L can be written as:

$$L = (y - XB)'(y - XB) \tag{41}$$

The right hand side can be expanded as:

$$L = y'y - (XB)'y - y'XB + (XB)'XB \tag{42}$$

$$= y'y - B'X'y - y'XB + B'X'XB \tag{43}$$

$$= y'y - 2B'X'y + B'X'XB \tag{44}$$

Now the trick is to find B which minimizes L. This can be found by partial differentiation of L with respect to B.

$$dL/dB = -2X'y + 2(X'X)B \tag{45}$$

Setting this partial differential equal to zero results in the normal equations in the estimation of B:

$$(X'X)B = X'y \tag{46}$$

Now solving for B results in the solution to the normal equations assuming that $(X'X)^{-1}$ exists (ie. the matrix is nonsingular):

$$B = (X'X)^{-1}X'y \tag{47}$$

These are called the normal equations in the estimation of B (28:68-69).

The general linear model can easily expanded from a first order model to a higher degree polynomial. In this research, a special characteristic of RSM in which the levels of the x variables are chosen evenly spaced will yield a special class known as orthogonal polynomials. These polynomials give some simplification to the computations and their desirable properties will be discussed later.

Factorial Experiments and Experimental Designs

In this research, factorial experiments will play a major factor in the estimation of the response polynomial. Factorial experiments are used when a researcher is interested in finding how a response, y, is influenced by certain combinations of inputs. A well-designed experiment can save the researcher both time and money. The combination of factor levels is combined in a design matrix, D:

$$D = \begin{vmatrix} x_{11} & x_{21} & \cdots & x_{k1} \\ x_{12} & x_{22} & \cdots & x_{k2} \\ x_{1n} & x_{2n} & \cdots & x_{kn} \end{vmatrix}$$

where the u^{th} row, $[\, x_{1u}, x_{2u}, \ldots, x_{ku} \,]$ represents one experimental run of the simulation (28:108).

The simplest factorial experiment for determining first-order effects (simple linear model) is the 2^k factorial design. This is k factors each at two levels. The two levels can be denoted as low and high and a convenient -1 and +1 convention can be used as notation of low and high factor levels, respectively. This corresponds to the transformation:

$$x_i = 2(E_i - X_i)/d_i \qquad (48)$$

where the E_i is the actual reading, X_i is the mean value between the low and high factor level, and d_i is the spacing between the low and high factor level (28:108).

When ±1 coding of the variables is used , the design columns are called orthogonal if the sums of their cross-products are zero. Also, two columns are orthogonal if their factor levels are balanced. Orthogonality is a desirable design property because: (1) the design matrix can conveniently display the levels in the experiment and (2) estimates of the main effects of the orthogonal factors are independent. Therefore, if one of the orthogonal factors has an effect, it cannot cause the other to appear to have an effect (37:254).

Using an orthogonal design to estimate a polynomial response surface helps to eliminate some underlying difficulties inherent in this type of estimation. Because an orthogonal design is used, the regression coefficients are uncorrelated, or:

$$\text{cov}\,|b_i| = s^2(X'X)^{-1} \tag{49}$$

This results in the covariance between any two coefficients being zero. Designs of this nature result in X'X being diagonal and thus, the coefficients of the model are uncorrelated (28:45-46). If X'X is not diagonal, then the model must be rewritten in the form:

$$\mathcal{E} = b_0 + b_1x_1 + b_2x_2 + b_{11}(x_1^2 - x_1^{2*}) + b_{22}(x_2^2 - x_2^{2*}) + b_{12}x_1x_2 \tag{50}$$

where $\mathcal{E}$ is the same dependent variable as y and x_1^{2*} and x_2^{2*} are the mean values of x_1^2 and x_2^2, respectively (28:50).

However, in this research a second-degree polynomial needs to be estimated. Although the simple 2^k design was useful in the preliminary stages of the experiment, a new factorial design must be introduced when the regression model is best estimated by a second order relationship (first

and second order terms are included in the model). This new design will be the 3^k factorial design in which k factors are varied at three levels. A similar transformation must be made to the input variables to give -1, 0, +1 values to the input combinations. The point (0,...,0) is known as the center of the design. Therefore, the 3^k design also falls into the class of orthogonal designs. One obvious disadvantage of the 3^k design is that for a many factor experiment, the number of design points can become unmanageable very quickly (ie. 4 factors = 81 design points). Also, full and fractional factorial designs are not very effective at estimating higher-order effects. This is because the number of experiments necessary is normally very much higher than the number of effects to be estimated. Also, these effects will be estimated with low precision. Usually, however, these effects are assumed to be negligible (28:126).

The basic goal of any experimental design is to find the factor level combination which optimizes a response and also that level which best explains the relationship between the factors and the response.

Response surface methodology uses experimental designs to find the optimal operating conditions from a set of input factors. These input factor combinations should produce an optimal response in the system. Brightman (1978) claimed that the use of response surface methodology in simulation experiments need to meet only two prerequisites: the measure of effectiveness be continuous and the input factors be quantitative (16:255).

Response Surface Methodology

Response surface methodology combines the best features of least-squares regression and experimental design to fit a response surface equation to a set of data over the experimental region of interest. In this

research, a second-order polynomial will be fit as the response surface. The experimental design needed to fit this second-order model must have at least three levels of each factor for the model's parameters to be estimated (27:462).

The most commonly used design for fitting a second-order model is the central composite design (CCD). This is the design choice for this research. This design consists of a 2^k factorial (coded $\pm$ 1) augmented with 2k axial points ($\pm$ alpha) and n center points (27:462). Six center points will be used for the central composite design in this research. Repeated observations at the center of the design can be used to estimate the experimental error and also to check the adequacy of the first-order model. This adequacy can be checked by comparing the average response at the corner points of the design with the average response at the center of the design. This difference will be a measure of the overall curvature of the surface (27:449-450).

Since all the second-order derivatives of this model must be estimated, it will be necessary to begin with a complete 2^k factorial design to fit a first-order model and then add design points to fit a second-order model. These extra points will be added to form a central composite design with alpha equal to one. This alpha value will be chosen so the design remains orthogonal. Therefore, all second-order derivatives may be calculated with equal precision. This will make the CCD very effective for analysis.

Canonical Analysis

The final goal of fitting the response surface function will be to determine the nature of the stationary point and the entire response system (28:72). This is simply a translation of the response function from the origin to the stationary point. New variables, w_i's, will be used instead of x_i's to express

the new system. The w_i's relate to the major axes of the new contour system.

The relationship between the x_i's and the w_i's is of great importance. If the stationary point is found to be outside the region of interest, the relationship between the variables can lead to areas where further experimentation may be more productive (28:73).

Statistical Tests

The only major statistical test this research effort will be concerned with will be a check of whether or not the model is a correct approximation of the actual system. One procedure often used is a test for lack of fit.

Remember that $e_i = Y_{observed} - Y_{estimated}$ is the residual at X_i. Calculation of the residuals yield valuable information about how the estimated model fails to explain the variation in the response variable Y. This is commonly referred to as bias error. If the model is correct, then the bias equals zero and the residuals and residual mean square can be used as an estimate of the variance, s^2, in the model (31:267-271).

If the model is incorrect, however, then the bias does not equal zero and the reisduals contain both random and systematic errors. These are commonly referred to as variance and bias error. The residual mean square can no longer be used as an estimate of the variance in the model.

In the case of simulation experiments, no prior estimate of the variance exists. In such a case, replications of the Y's and each value of X can be used to obtain an estimate of the variance. This estimate is known as pure error because only random variations can cause differences in the results. These differences will usually provide a reliable estimate of the variance in the model (14:35).

Therefore, the residual sum of squares with n_r degrees of freedom can be divided into two components: (1) pure error sum of squares from the repeated observations with n_e degrees of freedom and (2) lack-of-fit sum of squares with $n_r - n_e$ degrees of freedom. The pure error SS leads to s_e^2 or mean square due to pure error and estimates the variance in the model. The lack-of fit SS leads to MS_L or mean square due to lack of fit. MS_L estimates the variance in the model if the model is correct or the variance plus the bias if the model is incorrect (14:36).

Then, an F-ratio of MS_L/s_e^2 with 100(1 - alpha)% point of an F-distribution with $(n_r - n_e)$ and n_e degrees of freedom can be compared. The usual hypothesis for this test can be simply stated as:

H_0: a linear model is adequate

vs.

H_a: a higher order model would better represent the data

If the ratio proves significant then the model appears to be an incorrect representation of the system. If the ratio is insignificant then the model appears to be a correct representation and both the pure error and the lack-of-fit mean squares can be used as estimates of the variance in the model (14:37).

Draper (1981) describes the entire procedure as follows:

1. Fit the model with the analysis of variance table for the regression and residuals. Do not test the overall regression parameters yet.

2. Define the pure error sum of squares and divide the residuals as described above.

3. Do an F-test for lack of fit.

4. If lack of fit is significant, stop the analysis of the current model and search for another, more improved model.

5. If lack of fit is insignificant, recombine the pure error and lack of fit sum of squares into the residual and use this as an estimate of the overall variance in the model. Now do an F-test on the overall regression parameters (14:40).

Multiple Responses

The beauty of response surface methodology comes from the sequential process of forming a good-fitting second-order polynomial and then finding that its stationary point results in a point of maximum yield. However, this beauty quickly fades when it is found that the point of maximum yield is infeasible due to some cost constraint not considered in the analysis. In other words, a point of maximum yield will be found in some experimental region. Usually this experimental region covers the entire feasible region where the x_i's have some significance. A second response function, using the same response surface techniques, can be used to show some cost constraint as a function of cost considerations at each design point imposed on the original response process. Now the cost constraints can be observed simultaneously with the yield responses.

Proceeding blindly without consideration of the second response can also lead to further frustration if the analysis used to find the point of maximum yield was done without the prior knowledge of the cost constraint (26:189). To solve this dilemma, response surface methodology adopts itself well to multiple response systems. Contour plots of the response surface can show more than one region where the predicted response is at a satisfactory level. By combining this information with similar contours from a second response surface, a movement can be made to a region that approximates the optimal or close to optimal operating conditions. For experiments with a small number of factors, three or less, multiple responses can be effective by

superimposing the contour diagrams of the various response surfaces. Now the experimenter can visually determine the "best" operating conditions (28:167).

Multiple response surface analysis will be the basis of this research effort. Hopefully, by superimposing the minimum cost contours onto the maximum yield contours the actual operating conditions, both in terms of output and cost, can be visually displayed. It may very well be that the stationary point for the output and the stationary point for the cost lie within the same critical level. This would be totally optimal--maximum production and minimum cost. But if the critical levels do not intersect, the experimenter can see in which direction to move from the stationary point on the output surface to a region, according to the contour diagrams, where the output is "close enough" to a maximum but also the cost of production is low enough to meet the constraints of the system (21:67-68).

Characteristics of a Production Function

The basic definition of a microeconomic production function is that it "expresses the relation between a maximum amount of output and the inputs required to produce it; in doing so it describes the manner in which inputs co-operate with each other in varying proportions to produce any given output (10:26)."

Any function which shows the dependence of output, Y, on two factors of production, say labor, L, and capital, K, where: $Y = f(L,K)$ can be termed a production function. This is because the function relates output to input and also describes the relationship between the inputs.

The first and most basic criterion of any production function is that any increase in each input should have a positive effect on the output. In

microeconomic terms, the marginal products should be positive. This shows that the output, y, can be increased by increasing one of the inputs while holding the other inputs constant. This can be shown by:

$$dY/dL > 0 \text{ and } dY/dK > 0 \qquad (51)$$

These inequalities imply that the constant production curves are downward sloping. Thus, if both inputs have a positive effect on output, to keep output constant, if the amount of one input is increase, the amount of the other must simultaneously decrease. Therefore, the slope of the isoquant curve is negative (30:237).

A second criterion is that the marginal product should decrease when both inputs increase (30:238). This ensures that the critical point is a true maximum. For a true maximum to result, the output, y, should be decreasing for any change in the inputs away from the critical point. This can be shown by:

$$d^2Y/dL^2 < 0 \text{ and } d^2Y/dK^2 < 0 \qquad (52)$$

Another microeconomic term, isoquants, must be introduced here. The isoquants represent the marginal rate of substitution of labor for capital, or:

$$-(dK/dL) = (dY/dL)/(dY/dK) \qquad (53)$$

Here, the marginal rate of substitution of labor for capital decreases as labor is substituted for capital. The ratio of the marginal products will decrease with a rise in labor, given a certain amount of capital or a decrease in the amount of capital, given an increase in labor. It is often assumed that cost minimization requires the inputs to exhibit diminishing marginal productivity but what is actually required is a diminishing marginal rate of substitution of labor for capital (30:243-246).

The isoquants, when graphically depicted, show the substitutability of labor for capital at a constant level of output. Any point along the isoquant

describes the same output level. Thus, they can describe the measure of ease with which labor can be substituted for capital. This, as mentioned earlier, is known as the elasticity of substitution.

For the purpose of this research, these two simple principles are all the microeconomic knowledge necessary for discussing the response surface used as the "industry" production function. The resultant production function will be tested and analyzed to see if it meets the above criteria.

Description of the Simulation Model

To represent the methodology discussed in this research, a computer simulation model of a typical resource-allocation system was needed. Such simulation models are commonly used throughout the Air Force as an aid to decision makers when it is either too costly or infeasible to observe the actual system in real life. The simulation model does not necessarily have to be complex. The only requirement is that it is as accurate a representation of the real-life system as is possible.

One such simulation model was already available. It is a SLAM II (32) terminating simulation model which attempts to model the requirements needed to conduct a mission-effective thirty-day war. Four input variables, factors, are used in the model: number of aircraft, number of support kits, number of maintenance crews, and number of bombs. These four factors affect the number of sorties that can be flown during this thirty-day war.

The object of this simulation is obvious--maximize the number of sorties which can be flown during this thirty-day war subject to the levels of the four factors. One large assumption was made which is not always wise for the analyst to assume--the computer program used to run the simulation

model has been properly debugged so the model functions as closely to the system as intended (37:250).

The goal of this simulation will be to use the simulation generated output in order to estimate the parameters of the system. The replication method will be used with the simulation runs to generate the output data. This method has the advantage in that, given random seeds, the simple statistics are independent and identically distributed. These statistics are then quite useful for statistical analysis. Also, the replication method is the only recommended method for analyzing a terminating simulation (16:245).

For the purpose of this research, only two of the four factors will be varied: support kits and maintenance crews. This will make for a handy relationship to capital (support kits) and labor (maintenance crews) in the resultant production function. Thus, a microeconomic analysis can be done on the function if the appropriate conditions for cost minimization ,described on pages 6 through 9, are met throughout some region where the marginal products (f_1 and f_2) are positive. Throughout the simulation runs, the number of available aircraft will begin at 24 and the number of available bombs will begin at 288. The initial values of the model variables are quite important to the output of the system. These starting conditions should mirror the real-life system as closely as possible. Often times, it is helpful to use real-life data or a draw from a probability distribution fit to the real-life data (16:247).

Since this is a trace-driven simulation model (a simulation involving the use of historical data arriving in a single time series pattern), repeated use of the model will reduce the variation of the output. Thus by starting the different simulation runs at the different design points with the same common random number seeds (common streams), a similar variance

reduction will also take effect. Also, since alternatives will be tested in this simulation through the different input combinations, common streams will play an important role in reducing the variance of the output responses. Variance reduction techniques simply replace the original sampling procedure with one with the same expected value for the parameter estimates but of smaller variance (16:249).

This can be shown simply by:

$$\mathrm{Var}\,[X^{(1)} - X^{(2)}] = \mathrm{Var}\,[X^{(1)}] + \mathrm{Var}\,[X^{(2)}] - 2\mathrm{Cov}\,[X^{(1)},X^{(2)}] \qquad (54)$$

where $X^{(k)}$ is the sample mean for the alternative response k. Common streams will reduce the variance in the model since the Cov $[X^{(1)},X^{(2)}]$ should be positive resulting in an overall reduction in the model's variance (32:745).

For terminating simulations, run length is not an issue since the simulation ends when a specific event occurs. Thus, the desired level of precision for the experiment cannot be affected by the run length. Sample size or how many replications of the simulation to run, however, is an important problem in all simulations which are used to generate output data. Unfortunately, there is no solid rule for determining how many simulation runs to make. For the purpose of this research, the problem will be handled by collecting data and testing to see if it meets a specified criterion. If the test fails, then simply collect more data until the criterion is met (16:253).

The advent of computer simulation makes the process of repeat observations quite rudimentary. Following the RSM procedure described previously, the simulation model will be run initially with both factors at their low and high levels for two cycles with six cycles at the center point. This requires a total of fourteen runs of the simulation model. These responses will be statistically tested for lack-of-fit of the linear model. If such a lack-of-fit is significant, then only four more design points need to be

added or a total of eight additional simulation runs to achieve the full set of data required to properly fit a second-order polynomial.

III. Findings and Analysis

The subject of this response surface analysis is a resource allocation simulation that models the number of aircraft sorties that can be reasonably expected to be flown during a typical thirty-day war. The actual simulation code used in this research effort is listed in the Appendix. The overall objective is to fly the maximum number of sorties possible subject to the limited number of crews and support kits available. This is a typical resource allocation simulation in which the amount of inputs will vary and a response (y) will be measured; in this case, y represents the number of sorties flown.

Since this is a simple two-variable case with x_1 and x_2 representing the number of aircraft support kits and maintenance crews respectively, a complete factorial design was formed. In this case it will be a 2^2 factorial design with center points. That is, each factor in the simulation is input at its low and high levels with the center point being the midpoint between the low and high level. Since the purpose of this thesis is to describe polynomial production functions, the first step in any research will be to see if any curvature in the response function exists. The object at this point is simply to find out if a first-order model can adequately describe the system; that is, the response y as a function of x_1 and x_2 only with no crossproduct term and no quadratic effect. An analysis of variance of the simulated data will be the method used for significance testing and parameter estimation. This could be easily accomplished since the number of independent variables (2) is small. If the number of input variables increases, the complete factorial design procedure becomes increasingly complex. Obviously, in order to

produce a result that is both reasonable and feasible, some bounds must be placed on the region of interest. It will be assumed that no more than 10 support kits and 12 maintenance crews can be available at any one time. Minimum values for support kits and maintenance crews will be set at six each. Also, even though the numbers are discrete they will be used as though they were continuous in this model.

Experimental Designs

Some previous knowledge of the system is always important for the simulation model to be accurate and the analysis to be effective. In this case, previous results show that the response output operates over a region where the number of support kits, x_1, varies between 6 and 10 and when the number of maintenance crews, x_2, varies between 6 and 12. This range of design points should be large enough to minimize the average variance over the experimental region. Since the ultimate goal of this research will be to fit a second-order polynomial, guarding against curvature (ie. minimizing bias) will not be a concern (28:197-198). This experimental design will then be one which minimizes variance over the bounds on the region of interest for the analysis. Ranges were chosen which would be wide enough to enable detection between differences in the response while not exceeding the practical limits on the system and also minimizing the variance of the experiment. Now, the experiment can begin with a simple 2^2 design in this region

This design was the natural choice for this analysis for the following reasons: (1) The factorial design is an efficient method of experimentation. It can provide quick information on the effects of several variables. (2) The factorial design provides a measure of interaction between control variables

if it exists. A system which contains interaction will exhibit curvature in its response relative to changes in the levels of the independent variables. This would not be detected with one-at-a-time experimentation. (3) Curvature may also be detected by adding a center point to the design. The difference between the response at the center point and the mean for all the corner points is a measure of "lack of fit". Lack of fit refers to the linear model's inability to accurately represent the data. (4) The factorial design can later be expanded to provide an estimate of curvature if it exists. Thus, the experiment can proceed sequentially-first with a relatively simple experimental design, and later-if necessary-with a more complicated design (7:5-8).

Phase I of the Experiment

This region in the 2^2 design has a low value of 6 for both support kits and maintenance crews (control variables x_1 and x_2 respectively) and a high value of 10 for kits and 12 for crews. Throughout the remainder of this paper, kits may be used as an abbreviation for support kits and crews as an abbreviation for maintenance crews. The centerpoint is at (8,9). That is where the number of kits is 8 and the number of crews is 9. This is the midpoint of the low and high levels for each factor. Two variables, each at two levels, results in 2^2 or 4 treatment combinations. The center point added to the design will result in a total of 5 treatment combinations. Three runs at the center point and one run at each corner point will be one cycle of the experiment. Two cycles of the simulation will be made at each of these four design points and the center point--a total of 14 runs. The multiple runs at the center point will give the analyst an estimate of "lack of fit".

Common random number seeds will be used throughout this experiment to reduce the variance. A first-order model can be used to determine whether or not higher-order terms are needed in the response surface equation and thus further experimentation. The following figure shows the levels used to test the adequacy of the first-order model.

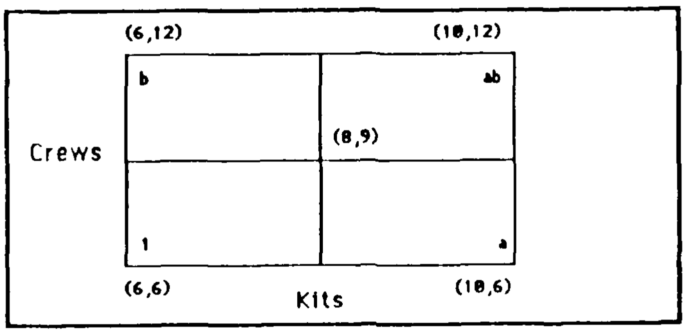

Fig. 1 Design Points for Fitting First-Order Model

Analysis of the First-Order Model

The experiments were repeated twice to provide an estimate of the experimental error. The size of the measured effects due to changes in the level of the control variables is compared with the experimental error. If the size of the effect is large relative to the experimental error, it is recognized that the change in response cannot be attributed to random chance. The effect is then considered a "real" effect.

The following table lists the design points, their corresponding coded variables, and the responses that resulted from the simulation runs at each design point. These data points will be used in fitting the first-order model.

Table I

Data for First-Order Model

Treatment	Natural Variables		Coded Variables		Response
	E_1	E_2	X_1	X_2	Y
(1)	6	6	-1	-1	1097
(1)	6	6	-1	-1	1096
a	10	6	1	-1	1146
a	10	6	1	-1	1147
b	6	12	-1	1	1098
b	6	12	-1	1	1024
ab	10	12	1	1	1125
ab	10	12	1	1	969
y_{01}	8	9	0	0	1147
y_{02}	8	9	0	0	1096
y_{03}	8	9	0	0	1120
y_{04}	8	9	0	0	1110
y_{05}	8	9	0	0	1071
y_{06}	8	9	0	0	1130

The results can also be conveniently displayed in a design table:

Table II

Coded Design Data for ANOVA

	(A) KITS 6	(A) KITS 10	$y_{i..}$
(B) CREWS 6	(1): 1097, 1096; total 2193	a: 1146, 1147; total 2293	4486
12	b: 1098, 1024; total 2122	ab: 1125, 969; total 2094	4216
$y_{.j.}$	4315	4387	8702 = $y_{...}$

This type of design with added center points is known as an augmented 2^2 factorial design (28:81). It is an efficient method of testing the adequacy of the simple first-order model for lack-of-fit (presence of cross-product and pure quadratic terms). The result will be presented in the following table:

Table III

ANOVA of Augmented 2^2 Factorial Design

Source	d.f.	Sum Squares
Total, Uncorr.	14	SSToTU
B_0	1	SS_0
B_1	1	SS_1
B_2	1	SS_2
Remainder	11	SS_{CP}
Cross Products	1	SS_Q
Pure Error	9	SS_{PE}

where SSTotU = y'y and SS_0 = y'y/14.

The contrasts for the linear coefficients can be summarized as:

Table IV

Treatment Combinations

	Treatment				
Effect	(1)	a	b	ab	L_i
B_1	-	+	-	+	L_1
B_2	-	-	+	+	L_2

where $SS_1 = L_1^2/8 = (72)^2/8 = 648$ and $SS_2 = L_2^2/8 = (-270)^2/8 = 8842.5$.

Now the parameters can be estimated by simple factorial estimation procedures (27:264):

$$b_0 = [(1) + (1) + a + a + b + b + ab + ab]/8 = 1099.43 \quad (55)$$

$$b_1 = [-(1) - (1) + a + a - b - b + ab + ab]/8 = 9 \quad (56)$$

$$b_2 = [-(1) - (1) - a - a + b + b + ab + ab]/8 = -33.75 \quad (57)$$

$$b_{12} = [(1) + (1) - a - a - b - b + ab + ab]/8 = -16 \quad (58)$$

An unbiased estimate of the error in the model, s^2, will be the pure error (SS_{PE}) plus the error accounted for in the repeat observations. In other words, the nine error degrees of freedom are a combination of (n_{CP} - 1) degrees of freedom for the six runs at the center point and four degrees of freedom for the replications at the design (corner) points (27:449-450).

The repeat observations at the center can be used to calculate an estimate of pure error as follows (14:34):

$$SS_{PE} = ((1147)^2 + (1096)^2 + \dots + (1138)^2) - (1147 + 1096 + \dots + 1138)^2/6 \quad (59)$$

$$MS_{PE} = 4044/5 = 808.8 \quad (60)$$

A popular check of the straight-line model is obtained by comparing the *average response at the four corner points in the factorial design*, $y_1 = 1087.75$, with the average response at the center point of the design, $y_2 = 1115$. If the design represents a curved surface, then $y_1 - y_2$ is a *measure of the surface's overall curvature. If B_{11} and B_{22}* are the coefficients of the "pure quadratic" terms x_1^2 and x_2^2, then $(y_1 - y_2)$ is an estimate of $B_{11} + B_{22}$ (27:450). Thus, an estimate of the pure quadratic term is:

$$\begin{aligned} B_{11} + B_{22} &= y_1 - y_2 \\ &= 1087.75 - 1115 \\ &= -27.75 \end{aligned} \quad (61)$$

The single-degree-of-freedom sum of squares associated with the contrast $(y_1 - y_2)$ is:

$$SS_{PC} = [n_1 n_2 (y_1 - y_2)^2]/(n_1 + n_2) \quad (62)$$

$$= [(8)(6)(-27.75)^2]/14$$

$$= 2545.93$$

where n_1 and n_2 are the number of points in the factorial portion and the number of center points in the design, respectively (27:450). Since:

$$F = SS_{PC}/MS_E \quad (63)$$

$$= 2545.93/449.33$$

$$= 5.67$$

which would be compared to $F_{.05,1,9} = 5.12$, there is a significant indication of a quadratic effect (27:450).

The single degree of freedom sum of squares for the cross product is (31:275):

$$SS_{CP} = SSTotU - SS_0 - SS_1 - SS_2 - SS_Q - SS_{PE} \quad (64)$$

$$= 16{,}955710 - 16{,}935{,}598 - 648 - 8842.5 - 4044 - 2545.93$$

$$= 4031.57$$

Comparing SS_{CP} to s^2 gives the following lack-of-fit statistic:

$$F = SS_{CP}/s^2 = 4031.57/449.33 = 8.97 \quad (65)$$

which would be compared to $F^*_{.05,1,9} = 5.12$. Clearly, interaction is significant. The following table shows the analysis of variance based of two cycles of the factorial design.

Table V

ANOVA for First-Order Model

Source	Sum of Squares	d.f.	Mean Square	F-Ratio
Linear Model	$SS_R = 9490.5$	2	4745.25	10.56
Kits	$SS_A = 648$	1	648	1.44
Crews	$SS_B = 8842.5$	1	8842.5	19.68
Lack of Fit	6577.5	2	3288.75	7.32
Crossproduct	3607.25	1	3607.25	8.03
Quadratic	2970.25	1	2970.25	6.61
Pure Error	$SS_E = 4044$	9	449.33	
B_0	16,935,598	1		
Total, Uncorr.	16,955,710	14		

$$\text{Prediction equation: } Y = 1087.75 + 9x_1 - 33.75x_2 \qquad (66)$$

where: $x_1 = (\#\text{kits}-8)/2$ and $x_2 = (\#\text{crews}-9)/2$.

The null hypotheses to be tested can be stated as: the crossproduct and quadratic terms are equal to zero, signifying no curvature in the response model. As one can see, there is enough information to reject the null hypothesis that neither the crossproduct term is equal to zero ($F^* = 8.03 > F = 5.12$) or the pure quadratic term ($F^* = 6.61 > F = 5.12$) is equal to zero. Also, the lack of fit can be tested with two degrees of freedom: one for the SS_{CP} and one for the SS_Q. Comparing this F-value = 8.97 to $F_{.05,2,9} = 4.26$, lack of fit is significant. This supports the above claim, so a move to a higher-order model is warranted.

Phase II of the Experiment

The following matrices are needed for the least-squares regression techniques involved in fitting the second-order response surface. The design matrix for this analysis with the added design points is:

Table VI

Design Matrix

D =

X_1	X_2	X_1X_2	X_1^2	X_2^2
-1	-1	1	1	1
-1	-1	1	1	1
1	-1	-1	1	1
1	-1	-1	1	1
-1	1	-1	1	1
-1	1	-1	1	1
1	1	1	1	1
1	1	1	1	1
1	0	0	1	0
1	0	0	1	0
-1	0	0	1	0
-1	0	0	1	0
0	1	0	0	1
0	1	0	0	1
0	-1	0	0	1
0	-1	0	0	1

The X matrix with the extra center points added to test lack of fit:

Table VII

X Matrix with Extra Center Points

X =

b_0	X_1	X_2	X_1X_2	X_1^2	X_2^2
1	-1	-1	1	1	1
1	-1	-1	1	1	1
1	1	-1	-1	1	1
1	1	-1	-1	1	1
1	-1	1	-1	1	1
1	-1	1	-1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	0	0	1	0
1	1	0	0	1	0
1	-1	0	0	1	0
1	-1	0	0	1	0
1	0	1	0	0	1
1	0	1	0	0	1
1	0	-1	0	0	1
1	0	-1	0	0	1
1	0	0	0	0	0
1	0	0	0	0	0
1	0	0	0	0	0
1	0	0	0	0	0
1	0	0	0	0	0
1	0	0	0	0	0

And the Y matrix with the results of the simulation runs corresponding to the levels of the x_i's is:

Table VIII

Y Matrix of Responses

$$Y = \begin{vmatrix} 1097 \\ 1096 \\ 1146 \\ 1147 \\ 1098 \\ 1024 \\ 1125 \\ 969 \\ 1074 \\ 1147 \\ 1133 \\ 1086 \\ 1124 \\ 1124 \\ 1122 \\ 1120 \\ 1147 \\ 1096 \\ 1120 \\ 1110 \\ 1071 \\ 1130 \end{vmatrix}$$

To fit a second-degree polynomial response function, each factor must be varied at three levels. Because of the care that was used in designing the original 2^2 factorial experiment, to obtain a complete 3^2 factorial experiment requires simulation runs at only four additional design points: (0,1),(0,1),(1,0), and (-1,0). This is simply the addition of midpoint levels to each design point and is shown in the following figure.

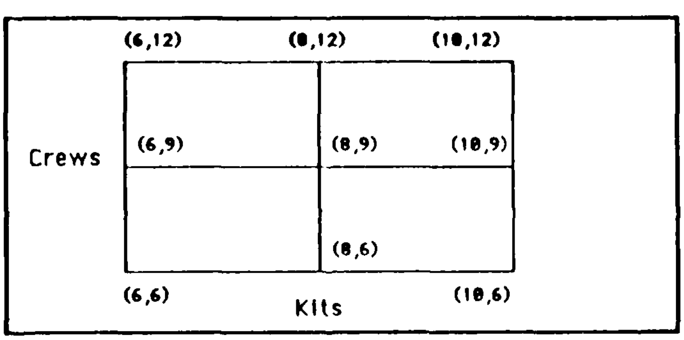

Fig. 2 Design Points for Fitting Second-Order Model

Analysis of Second-Order Model

The new data to be used in fitting the second-order response surface with the eight new design points is listed in the following table.

Table IX

Data for Second-Order Model

Natural Variables		Coded Variables		Response
E_1	E_2	X_1	X_2	Y
6	6	-1	-1	1097
6	6	-1	-1	1096
10	6	1	-1	1146
10	6	1	-1	1147
6	12	-1	1	1098
6	12	-1	1	1024
10	12	1	1	1125
10	12	1	1	969
8	6	0	-1	1122
8	6	0	-1	1120
8	12	0	1	1124
8	12	0	1	1124
6	9	-1	0	1133
6	9	-1	0	1086
10	9	1	0	1074
10	9	1	0	1147
8	9	0	0	1147
8	9	0	0	1096
8	9	0	0	1120
8	9	0	0	1110
8	9	0	0	1071
8	9	0	0	1130

Now the levels of the x_i's which maximize the predicted response must be found. This critical point, if it exists, will be the set of x_1 and x_2 such that the partial derivatives $dy/dx_1 = dy/dx_2 = 0$. This point is called the stationary point. This stationary point could be any one of three possibilities: (1) a point of maximum response, (2) a point of minimum response, or (3) a saddle point. The canonical analysis which will be used to analyze the fitted second-order response will also describe the nature of this stationary point (28:70).

A general solution for the stationary point may be easily obtained. The second-order model can conveniently be written in matrix notation:

$$y = B_0 + \mathbf{x}'b + \mathbf{x}'B\mathbf{x} \tag{67}$$

Here, b is a (k X 1) vector of the first-order regression coefficients and B is a (k X k) symmetric matrix whose diagonal elements are the pure quadratic coefficients (B_{ii}) and whose off-diagonal elements are one-half the mixed quadratic coefficients (B_{ij}, $i \neq j$) (28:69). The derivative of y with respect to the vector $\mathbf{x}$ and set equal to zero is:

$$dy/d\mathbf{x} = b + 2B\mathbf{x} = 0 \tag{68}$$

The solution for $\mathbf{x}$ is known as the stationay point, $\mathbf{x}_0$:

$$\mathbf{x}_0 = -1/2\ B^{-1}b \tag{69}$$

Also, but substituting this back into the original matrix equation, the predicted response at the stationary point can be found:

$$y_0 = B_0 + 1/2\ \mathbf{x}_0'b \tag{70}$$

The stationary point can be described further by transforming the fitted model into a new coordinate system with the stationary point, $\mathbf{x}_0$, being the origin. The axes of this new system are then rotated until they are parallel to the principal axes of the fitted response surface. The results can be shown by the fitted model:

$$y = y_0 + \text{lambda}_1 w_1^2 + \text{lambda}_2 w_2^2 \tag{71}$$

where w_1 and w_2 are the transformed variables and lambda_1 and lambda_2 are constants (28:73). This is known as the canonical form of the model where the lambda_i are simply the eigenvalues (characteristic roots) of the B matrix.

Now the characteristics of the response surface can be found from both the value of the stationary point and the signs and magnitudes of the

eigenvalues. Assuming that the stationary point is within the experimental region, if all the eigenvalues are positive, then a point of minimum response has been reached; if all the eigenvalues are negative, then a point of maximum response has been reached; and if the eigenvalues are of mixed signs, then a saddle point results. Also, the response surface is steepest in the w_i direction in which the corresponding eigenvalue is the greatest (28:75).

A second-order response model is fit by the method of least squares to the coded data to give the following response equation with appropriate t-ratios in parentheses:

$$\begin{array}{l} y = 1121.26 + 6.167x_1 - 22.00x_2 - 20.66x_1^2 - 8.16x_2^2 - 16x_1x_2 \\ \quad\;\; (74.77) \quad\; (0.52) \quad\;\; (-1.84) \quad\; (-1.12) \quad\;\; (-1.09) \quad\; (-0.44) \end{array} \tag{72}$$

The analysis of variance table follows:

Table X
ANOVA Table for Second-Order Response Surface

Regression	DF	Type I SS	F-Ratio	Prob
Linear	2	6264.333	1.83	0.192
Quadratic	2	3180.981	0.93	0.415
Crossproducts	1	2048.000	0.05	0.290
Total Regression	5	11493.314	0.30	0.296

Residual	DF	Sum Square	Mean Square	F-Ratio	Prob
Lack of Fit	3	4624.140	1541.380	0.882	0.476
Pure Error	13	22722.000	1747.846		
Total Error	16	27346.140	1709.134		

Lack of fit is not significant and regression is significant, however the poor F-values conclude that although a second-order response function will adequately approximate the response surface, a better polynomial could

approximate the true surface. A simple look at the response function's high intercept term causes some concern. This equates with zero kits and zero crews yielding 1121 sorties. This is not a very likely phenomenon. A new least squares regression will have to be done to derive a better response equation. But first, a preliminary look at an economic interpretation of this first response equation (production function) may shed some light on the purpose of this research.

Since this is still an unconstrained function, simple first- and second-order conditions (Chapter I) are all that is required to test for a maximum value over the region of interest. Considering the function:

$$y = f(x_1,x_2) = 1121.26 + 6.167x_1 - 22x_2 - 20.66x_1^2 - 8.16x_2^2 - 16x_1x_2 \quad (73)$$

The first order conditions for a maximum are shown as:

$$f_1 = 6.167 - 41.32x_1 - 16x_2 \quad (74)$$

$$\text{or } f_2 = -22 - 16.32x_2 - 16x_1 \quad (75)$$

This leads to the second-order partial derivatives:

$$f_{11} = -41.32 \quad (76)$$

$$f_{22} = -16.32 \quad (77)$$

$$\text{and } f_{12} = -16 \quad (78)$$

Sufficient conditions for a true maximum are that f_{11} and $f_{22} < 0$ and that $f_{11}f_{22} - f_{12}2 > 0$. These conditions are met since both second partial derivatives are less than zero and $(-41.32)(-16.32) - (16)^2$ is greater than zero. The above response function can be called a concave function and contains a true maximum point.

Setting both eqs (74) and (75) equal to zero result in $x_1 = 1.08$ and $x_2 = -0.2893$. Now these results must be entered back into the transformation eq (61) to yield the results: E_1 = # kits = 10 and

E_2 = # crews - 8. Also, $(f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2) < 0$ so conditions for cost minimization are possible. Therefore, the results to the transformation above will be a least-cost combination of inputs.

Since the condition where zero inputs will result in some positive output, a new least squares regression was done on the data however this time an intercept term of zero was forced upon the data. This makes empirical sense since zero inputs should result in zero outputs if the function is to be used as a true production function. The resulting response equation with the intercept term omitted and appropriate t-ratios in parentheses is:

$$\underset{}{y} = \underset{(4.945)}{217.29x_1} + \underset{(1.590)}{62.11x_2} - \underset{(-3.14)}{10.62x_1^2} - \underset{(-0.82)}{1.70x_2^2} - \underset{(-2.17)}{4.76x_1x_2} \qquad (79)$$

The first and most important hypothesis to test here is that the intercept term is indeed equal to zero. This is proven since the t-ratio for this test is equal to 1.586 with an equivalent p-value of 0.1312. This results in the null hypothesis not being rejected so the intercept term in the above function can be assumed equal to zero. A second important point is that both the x_1 and x_2 coefficients be positive. This supports the utility theory that more is better or any increase in the amount of x_1 or x_2 will result in a positive increase in the output response, y. If either coefficient carried a negative sign then the function could not be a true production function since the producer would not want any more of that input (even if it were given away free of charge) since any increase in that input would result in a decrease in the response, y (30:93-100).

Again, this new production function can be analyzed for cost minimization conditions. Both marginal products (f_1 and f_2) are positive and both of the second partial derivatives (f_{11} and f_{22}) are negative. However,

since the crossproduct term is significant and is negative, the second-order condition must be tested. Again, $(f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2) < 0$ so the conditions for cost minimization are possible. A range then where the marginal products are positive and the second-order conditons exist can be found in order to use this response surface as a production function.

The analysis of variance table for this least-squares regression follows.

Table XI
ANOVA for Second-Order Response Surface
with Test for No-Intercept

Source	DF	Sum Square	Mean Square	F-Ratio	Prob
Model	4	6745.046	1686.261	0.893	0.489
Error	17	32094.409	1887.906		
C Total	22	38839.455			
		Parameter Estimates			
Variable	DF	Estimate	Error	T-Ratio	Prob
Intercept	1	0	.	.	
X_1	1	217.29	43.94	4.945	0.001
X_2	1	62.11	39.06	1.590	0.130
X_1^2	1	-10.62	3.38	-3.140	0.006
X_2^2	1	-1.70	2.09	-0.817	0.426
X_1X_2	1	-4.76	2.19	-2.171	0.044
Restrict*	-1	8.38	5.28	1.586	0.131

* Intercept term can be set equal to zero

This second equation looks more like a true production function. Both x_1 and x_2 coefficients are positive and much larger than their squared terms. First- and second-order sufficient conditions for a true maximum since $(f_{11})(f_{22}) - f_{12}^2 > 0$ or $((-21.24)(-3.40) - (-4.76)^2) > 0$ are met. Also conditons for cost minimization exist over a certain region where realistic input values

can be found. Again, this is just a simple unconstrained function. Also, a complexity arises here in the two-variable case in that movements through the design points are not solely in the x_1 or x_2 direction. The second-order partial derivatives do not supply enough information on how the slope is changing through the critical point. Sufficient conditions must also be placed on the cross-partial derivative (f_{12}) to ensure that the response is decreasing through the critical point. Hence, the second-order partial derivatives must be sufficiently large to counterbalance any "bad" effects caused by the cross-partial derivatives. In other words, as the critical point falls in either the x_1 or x_2 direction any movements in the x_1x_2 direction can be compensated for (30:65-66). This is the reason that it is important to test whether or not the cross-product term in the function is equal to zero.

Thus, the equation:

$$y = 217.29x_1 + 62.11x_2 - 10.62x_1^2 - 1.70x_2^2 - 4.76x_1x_2 \qquad (80)$$

can be used as the production function for the simulation model.

Canonical Analysis

Using the results of the original least squares run, a canonical analysis can now be performed.

$$b = \begin{vmatrix} 6.17 \\ -22 \end{vmatrix} \qquad B = \begin{vmatrix} -20.66 & -8 \\ -8 & -8.16 \end{vmatrix} \qquad (81)$$

The stationary point is then:

$$x_0 = -1/2\, B^{-1}b \qquad (82)$$

$$= -1/2 \begin{vmatrix} -0.078 & 0.076 \\ 0.076 & -0.198 \end{vmatrix} \begin{vmatrix} 6.17 \\ -22 \end{vmatrix} = \begin{vmatrix} 1.08255 \\ -2.40998 \end{vmatrix}$$

Thus, the stationary point, $x_{1,0}$ and $x_{2,0}$, is equal to 1.08255 and -2.40998 respectively. Now these values can easily be converted back into their natural variables remembering the original transformations:

$$1.08255 = (E_1 - 8)/2 \text{ and } -2.40998 = (E_2 - 9)/3 \tag{83}$$

Now the natural variables are: $E_1 = 10.1651$ or 10 and $E_2 = 1.77006$ or 2. Remember that although the model uses these variables as continuous to make any sense of the results, they must be used as though they are discrete. In other words, the combination of kits and crews which is the point of maximum sortie generation is that of 10 kits and 2 crews. Obviously, in this crude example, to maximize the opportunity in the thirty day war, the number of kits at hand is much more valuable to the war effort than the number of crews.

Also, the maximum response at this stationary point, y_0, can be calculated by substituting back into $y_0 = B_0 + 1/2\, x_0' b$,resulting in, $y_0 = 1151.11$ or 1151. Now the response is telling the analyst that the maximum sortie generation he can expect from the input combination of 10 support kits and 2 maintence crews is 1151.

To further describe the stationary point, the canonical form may be obtained from the roots of the equation:

$$| B - \text{lambda } I | = 0 \tag{84}$$

The roots are known as the eigenvalues, lambda_1 and lambda_2.

$$\begin{vmatrix} -20.66 - \text{lambda} & -8 \\ -8 & -8.16 - \text{lambda} \end{vmatrix} = 0 \tag{85}$$

This now can be simplified to:

$$\text{lambda}^2 + 28.8158 \text{ lambda} + 104.525 = 0 \tag{86}$$

Using the quadratic formula, the roots of this quadratic are

$lambda_1 = -4.25592$ and $lambda_2 = -24.5599$. Therefore, the model's canonical form may be written as:

$$y = 1151.111 - 4.25592w_1^2 - 24.5599w_2^2 \qquad (87)$$

Since both $lambda_1$ and $lambda_2$ are negative, it can be concluded that the stationary point is a point of maximum response (27:453-454).

Since it is impossible to operate this system at the stationary point because the factor combination of $E_1 = 10$ and $E_2 = 2$ result in less than the minimum required number of crews, the decision maker may wish to move away from the stationary point to a point where E_2 is in the region of interest, but without sacrificing large amounts of sorties. If not, the system may be run with a combination of 10 kits and 6 crews (the minimum number of crews) since the number of crews is minor to the number of kits in producing a maximum of sorties, but this will not be the optimal solution in terms of cost to be mentioned later. To do this, it is necessary to find the relationship between the w_i's and the x_i's. This will relate the canonical variables back to the design variables. Looking at the response surface contours of the model, the response surface is less sensitive to sortie loss in the w_1 direction (the smaller of the two eigenvalues). Now, points in the (w_1, w_2) space must be converted to points in the (x_1, x_2) space.

The relationship between the w_i's and the x_i's may be described through a (k X k) orthogonal matrix, M (27:458-460). The columns of M are the normalized eigenvectors associated with the previously calculated eigenvalues. The x variables can be related to the w varaibles by the equation:

$$w = M'(x - x_0) \qquad (88)$$

If m_i is the i^{th} column of the matrix M, then m_i will solve the equation:

$$(B - \text{lambda } I)\, m_i = 0 \tag{89}$$

Using the previous results:

$$\begin{vmatrix} -20.6579 + 4.25595 & -8 \\ -8 & -8.15789 + 4.25595 \end{vmatrix} \begin{vmatrix} m_{11} \\ m_{21} \end{vmatrix} = 0 \tag{90}$$

or:

$$-16.40195 m_{11} - 8 m_{21} = 0 \tag{91}$$

$$-8 m_{11} - 3.90194 m_{21} = 0 \tag{92}$$

The normalized solutions to these simultaneous equations must result in $m_{11}^2 + m_{21}^2 = 1$. But, there is no unique solution to this. One preferred method is to let $m_{21}^* = 1$ and solve for m_{11} and then normalize this solution. With $m_{21}^* = 1$, $m_{11}^* = -0.4877$. Normalizing this, divide both m_{11}^* and m_{21}^* by $[(m_{11}^*)^2 + (m_{21}^*)^2]^{1/2} = [(-0.4877)^2 + (1)^2]^{1/2} = 1.11259$. Now, the normalized solutions are:

$$m_{11} = m_{11}^* / 1.11259 = -0.4877/1.11259 = 0.43835 \tag{93}$$

$$m_{22} = m_{22}^* / 1.11259 = 1/1.11259 = 0.89880 \tag{94}$$

This is the first column of the matrix M.

Next, using $\text{lambda}_2 = -24.5599$, the above procedure is repeated. The result is $m_{12} = 0.898785$ and $m_{22} = 0.438389$. This is the second column of the matrix M.

$$M = \begin{vmatrix} -0.43835 & 0.898785 \\ 0.89880 & 0.438389 \end{vmatrix} \tag{95}$$

and the relationship between the w variables and the x variables can be found by:

$$\begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = \begin{vmatrix} -0.43835 & 0.898785 \\ 0.89880 & 0.438389 \end{vmatrix} \begin{vmatrix} x_1 - 1.08255 \\ x_2 + 2.40998 \end{vmatrix} \tag{96}$$

which expands to:

$$w_1 = -0.43835\,(x_1 - 1.08255) + 0.898785\,(x_2 + 2.20998) \qquad (97)$$

$$w_2 = 0.89880\,(x_1 - 1.08255) + 0.438389\,(x_2 + 2.20998) \qquad (98)$$

Now, additional exploration of the response surface can easily be made in the region of the stationary point by finding points in (w_1, w_2) space to take the observations and then convert these to $(x1, x_2)$ space using the above equations. New simulation runs can easily be made at these new design points (27:460).

Phase III of the Experiment

Just as the high point on the sortie production contour represents a maximum yield, so does a low point represent a minimum yield. The true beauty of this entire methodology lies in this one basic assumption: if one can predict a region of maximum yield for a production process to operate within then by transforming the y_i's (responses) at the design variables into some sort of marginal cost response as a function of the combination of control variables, cost contours of the system can be plotted over the region. Again, simple RSM techniques would apply and if a minimum point does exist in the region of interest, this would represent an area of production where cost minimization could occur.

Now a multiple response function has been developed. On one contour plot lies the region of maximum output and on the second, the region of minimum cost. By simply overlaying the cost contours onto the output contours, many interesting and useful pieces of information can be studied (28:167-168). By inspection, the direction in which the producer would have to move to improve both output and costs of production could be seen simultaneously. If the cost and output contours intersect, then the producer

is operating at economically optimal conditions (minimum cost and maximum production). If they do not intersect, the direction of movement is readily at hand. Now, the potential risk of altering the input combination in an attempt to reduce costs is limited. The impact of an addition or reduction of one uni. of input to the cost of production is readily apparent.

Design of the Cost System

By simply giving some cost to each of the inputs and combining these costs throughout the experimental region to give some output cost function would only result in a response surface that is a rising plane. This is because the lowest cost combination would be in the lower left corner of the region and the highest cost combination would be in the upper right corner. Obviously, this would not be helpful to finding a minimum cost contour.

A method was designed based on the incremental cost of each input combination in the following manner (21:67). Each sortie flown was given an arbitrary cost of $1000 dollars. So each response y was multiplied by 1000. Now the total cost of the input combination (ie. $P_1 * 6$ kits + $P_2 * 6$ crews = Cost of the Inputs) was divided by this new response constant to give a new response variable which would be the incremental cost of adding the additional kits and/or crews to produce that number of sorties. The resulting data table was then formed:

Table XII
Data for Cost-Minimization Model

Natural Variables		Coded Variables		Response
E_1	E_2	X_1	X_2	$Y
6	6	-1	-1	0.0000109
6	6	-1	-1	0.0000109
10	6	1	-1	0.0000139
10	6	1	-1	0.0000139
6	12	-1	1	0.0000163
6	12	-1	1	0.0000175
10	12	1	1	0.0000195
10	12	1	1	0.0000227
8	6	0	-1	0.0000125
8	6	0	-1	0.0000125
8	12	0	1	0.0000177
8	12	0	1	0.0000177
6	9	-1	0	0.0000132
6	9	-1	0	0.0000130
10	9	1	0	0.0000177
10	9	1	0	0.0000166
8	9	0	0	0.0000140
8	9	0	0	0.0000155
8	9	0	0	0.0000150
8	9	0	0	0.0000153
8	9	0	0	0.0000159
8	9	0	0	0.0000149

The same response surface procedure as before was used on this data set. The key to this methodology is that no new experimentation is involved. A regression analysis was performed on the cost data and a new prediction equation was formed. Lack of fit was insignificant. Now the canonical analysis was performed around the stationary point ($x_{1,0}$ = 1.599, $x_{2,0}$ = -10.361) resulting in two positive eigenvalues of (4.328, 9.436). Thus, a point of minimum cost does exist in the experimental region. These are used to generate new cost contours. They are similar in appearance to the output contours generated previously. These cost contours can be plotted

and then compared to the output contours to see if the region of maximum output intersects with the region of minimum cost.

The point of minimum cost can be compared to the point of maximum output. This dual output-cost response leads to many useful pieces of information. If the region of maximum output intersects with the region of minimum cost than the "industry" is operating at economically optimal conditions. If these regions do not intersect, however, then the direction in which the "industry" must move to reduce its cost is readily apparent. This would obviously result in some reduction in output, but now the decision maker can visually observe the impact of output reduction caused by a reduction in costs (21:68).

A serious restriction must be placed on the application of this method. The prices of the kits and crews that result in a total cost constraint may be infeasible for the actual operation. One must be sure that the number of kits and crews that the response surface optimizes can be attained for a certain budget constraint. A region exists where certain prices of the inputs will not permit cost minimization to occur.

Alternative Method

A simple budget constraint mentioned in Chapter I can be imposed on the maximum yield response system through a simple cost constraint:

$$C = P_1x_1 + P_2x_2 \tag{99}$$

where P_1 and P_2 are the prices of the kits and crews, respectively, and C is the resulting total cost of the inputs. Plotting a response surface of this function is an easy task since the responses are simply the sum of the input

factors at each design point multiplied by their price. The resulting response surface will be a rising plane of costs from the lowest combination (P_1*6 kits + P_2*6 crews) to the largest combination (P_1*10 kits + P_2*12 crews).

These budget constraints can then be imposed on the response surface of the maximum yield and a feasible region where the system would operate can easily be seen. Each response contour can be treated as a level of constant output and the response contours treated as isoquants. The system will only operate where the isoquants were concave toward the origin. These isoquants will show the required combination of inputs that will produce the given level of output.

The slope of the isoquant will show the marginal rate of technical substitution (RTS) or how one input may be traded for another while holding the output constant. This rate of technical substitution will be equal to the ratios of the marginal products. This ratio will give the isoquants their negative slopes since both the marginal products and the rate of technical substitution must be positive, the negative of the RTS will be the slope of the isoquant. Finally, the isoquants will exhibit diminishing rates of technical substitution. This simply implies that the more crews that are used the harder it would be to substitute crews for kits for a given level of output (30:243-246).

IV. Conclusions and Recommendations

Restrictions on the Polynomial Response Function

In order to use a second-order polynomial as a production function, several restrictions on the function must be made so that it behaves as a production function an can be used in an economic analysis. Take any generic second-order polynomial:

$$y = b_0 + b_1x_1 + b_2x_2 + b_{11}x_1^2 + b_{22}x_2^2 + b_{12}x_1x_2 \tag{100}$$

to be used as a production function. The production process can be simply shown in the following figure where Y is the output of the production process and is a function of X, the combination of the two inputs used to produce Y.

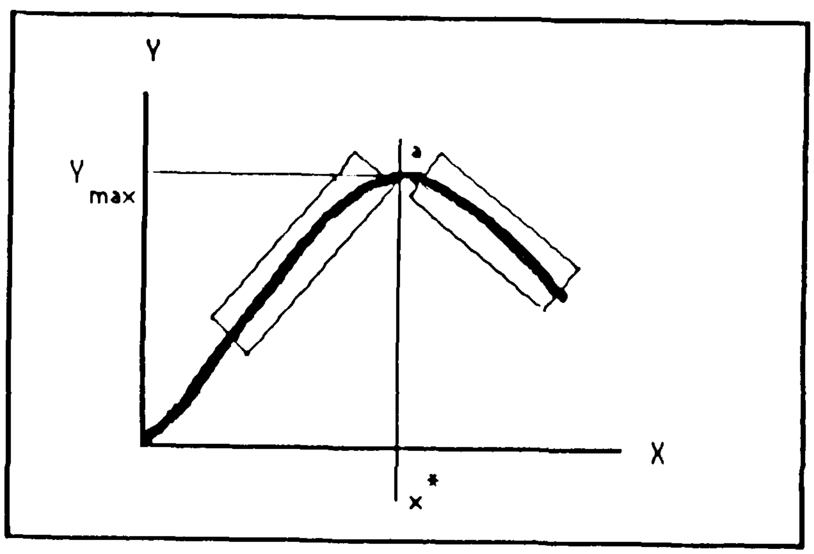

Fig. 3 Plot of Output Process

The point a is the point of maximum output, Y_{max}, that is produced at the combination of the X's at x^*. Obviously, the production process would like to be operated at the combination x^* to maximize its output. This point may only be reached if the budget constraint imposed on the system is large

enough to allow for the x^* combination of inputs. The second-order polynomial can reveal by simple inspection whether the production process is operating in a region where it is reasonable to expect a maximum output. The coefficients of the linear terms, x_1 and x_2, must be positive. The reason for this is that the positive sign indicates that they contribute positively to the output production. The experimenter then must take great care to ensure that the experiment is run in the region to the left of point a. Normal competitive industries would not even operate in the region to the right of point a (negative coefficients of x_1 and x_2). Here it is obvious that any increase in the amount of input would result in a decrease in the amount of output. No competitive firm would add any amount of input in this region even if the input were free.

However, a non-profit organization like the Air Force could operate in this region since it is not in the business of maximizing profits. So when a non-profit organization is modeled as a competitive, profit-maximizing firm, care must be taken to ensure that the modeling is done in the region to the left of point a to ensure that the resulting polynomial can be used as a production function.

This same care must be taken when computer simulations, as in this research, are used to model the input-output process. The experimenter must ensure that the experimental region covers the area where the marginal products are positive. This will ensure that the modeling is being done in the feasible region to the left of point a. It was shown in this research that simulation models can clearly lead the experimenter into a region where the marginal products are negative.

Another important restriction is that the second order condition:

$$f_{11}f_{22} - f_{12}^2 > 0 \qquad (101)$$

must be met in order that the function be concave and allow for an output maximization. The law of diminishing returns restricts the second partial derivatives, f_{11} and f_{22}, be negative. A relative maximum is an important point only for a second-order polynomial production function. A Cobb-Douglas or a CES production function never pass through a relative maximum point. Only for a polynomial production function does there exist an exact combination of inputs which will maximize the output process, if the imposed budget constraint allows that input combination to be attained.

Now the important point of testing for the existence of the cross-product term is echoed once again. If the cross-product term in the second-order response function can be proved insignificant and eliminated from the function, then the second-order condition will always be met since f_{12} will equal zero. If, however, the cross-product term cannot be eliminated from the function, then the second-order condition must be tested to ensure that the second-order polynomial does indeed behave as a production function and cost minimization is possible.

Analysis of Yield and Cost Responses

Now that the entire analysis needed for this research effort is complete, it is time to analyze the yield response function graphically. This can be accomplished by plotting the contours. Each contour represents a response level for a given amount of the inputs x_1 and x_2. Referring back to the canonical analysis done in Chapter 3, the eigenvalues not only determined that the response surface yielded a maximum response but also could be used to determine the shape of the contour surface. If both eigenvalues were equal then the response surface would be concentric circles around the critical point. However, in this instance, the second eigenvalue is much

larger than the first. Therefore, the response surface will be elongated in the w_1 direction. This corresponds to a greater loss in response in the w_2 direction or visually falling more quickly down the hill away from the critical point in the w_2 direction. The response surface is plotted in the following figure.

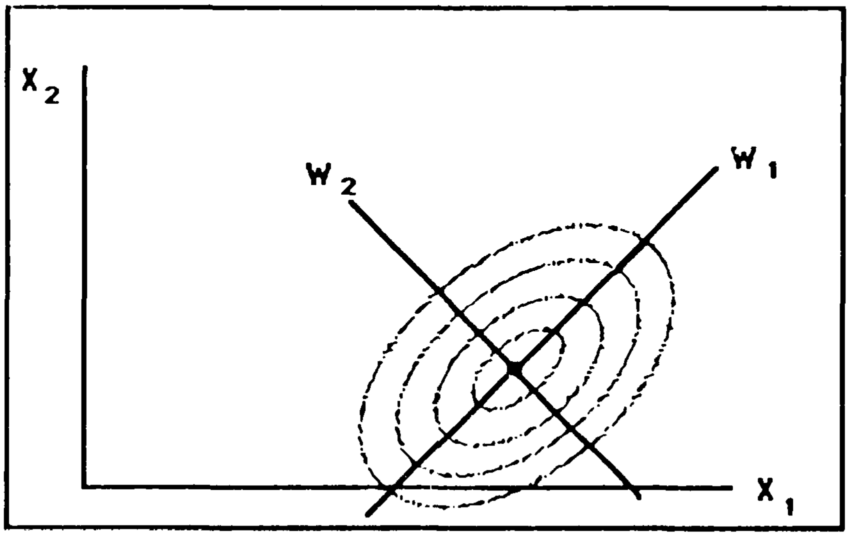

Fig. 4 Contour Plot of Maximum Yield

The response surface is tilted in the northeast direction due to the values of the eigenvectors found in the canonical analysis. The center point (critical point) of the response surface corresponds to a yield level of 1151 sorties. The number of kits and crews needed to produce this sortie output can be directly taken from the x_1 and x_2 values on the axes.

The cost responses are plotted on a separate graph in the same manner. This response surface reveals the minimum cost contours developed from different input combinations of x_1 and x_2. This time, the center point (critical point) represents the minimum cost combination of x_1 and x_2. Obviously, moving away from the center point or up the hill will result in increasing costs. Moving in the w_1 direction will increase the cost of the

system more quickly than movement in the w_2 direction. This increase in costs in the w_1 direction also corresponds to increasing input combinations of kits and crews. This revelation is intuitively obvious. The cost response surface is plotted in the following figure.

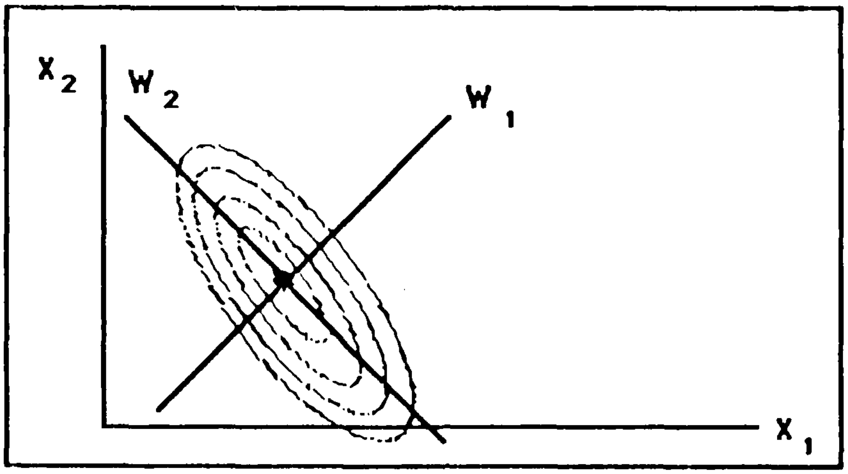

Fig. 5 Contour Plot of Minimum Cost

Analysis of Multiple Response System

The entire purpose of this research effort was to show that by representing a production process with a second-order polynomial response surface techniques could be used to analyze the production process. After estimating the function, plotting the resulting response function is an excellent way to show if a maximum output response is attainable in the region of interest. Also, the tradeoff between the input factors can be shown. Now if the production process is unconstrained to cost considerations then the single contour plot of the maximum yield region is the end to the study. But very few, if any, production processes do not consider costs and try to minimize them.

This constrained optimization problem can be very easily solved mathematically using Lagrange multipliers to find the least-cost

combinations of inputs. However, if the second-order polynomial function contains an interaction term this technique does not have a closed-form solution. Also, the least-cost combination of the inputs is found.

But often times, management wishes to look at the present operational process and see if it can reduce costs. This is where response surface methodology can be employed successfully. RSM can enable the analyst to graphically plot the production process. Not only can the input combination to yield the maximum response be shown, but also the reduction in output that results when tradeoffs in the input combinations are made.

The constrained optimization problem can be handled easily as a multiple response surface problem regardless of whether or not there is an interaction term in the production (response) function. By plotting both the output contours and the cost contours on the same graph, the constrained optimization problem can be shown visually. This is shown in the following figure.

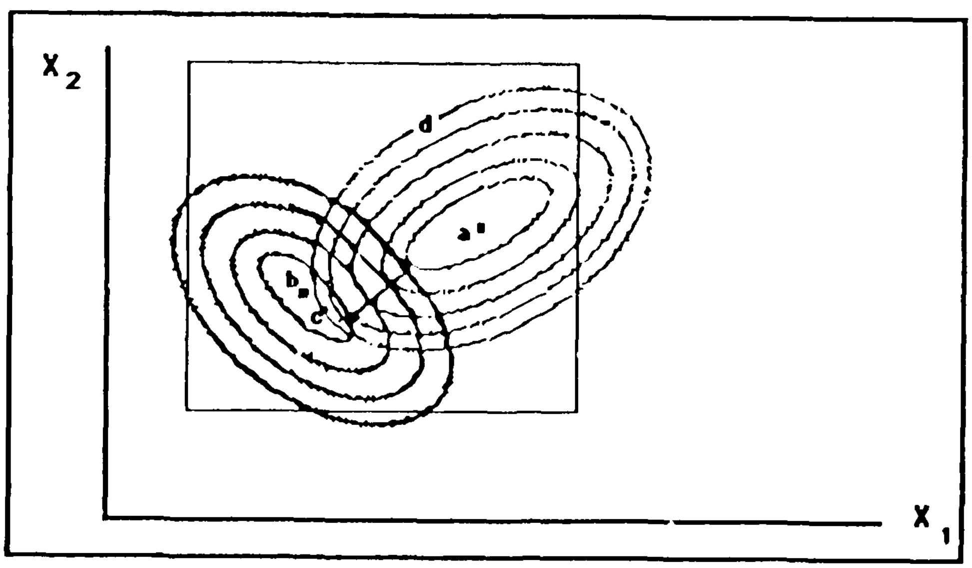

Fig. 6 Contours of Yield & Cost Responses

The square on the graph represents the feasible region of interest --(6 le. kits le. 10), (6 le. crews le. 12)-- used in this research effort. Point a is the point of maximum output and point b is the point of minimum cost. If the two regions around points a and b intersected, then the production process would be totally optimal--output maximization and cost minimization. *Obviously, this process is not totally optimal.* However, a feasible alternative may be reached. The locus of points along the path from a to b will take the production process to areas of less cost. The points along this path are the points where the slopes of the cost and yield contours are equal. Now the decision maker can see the reduction in yield (moving from one yield contour down to the next) as the move is made from point a to a region of lower cost. Point c would be the level of output that would minimize costs for this particular production process since that levels of response intersects the region of cost minimization. Although point d is located on the same yield contour (remember each contour represents the same output level) the producer would never want to operate at this level because the same output level could be achieved at point c with less amounts of inputs.

Remember the serious limitations on this type of multiple response analysis outlined in Chapter III. The alternative method discussed in the previous chapter will be discussed in the following section.

Analysis of the Alternative Method

The following figure illustrates how the maximum output response contours can be analyzed to show the region where the production process must be operated. The diagonal lines leading to the point of maximum output describe the feasible region for production. The response contours represent the increasing levels of ouput, up to the point of maximum output.

Each response contour represents a constant output level. In this way, the response contours may be thought of as the isoquants of the production process.

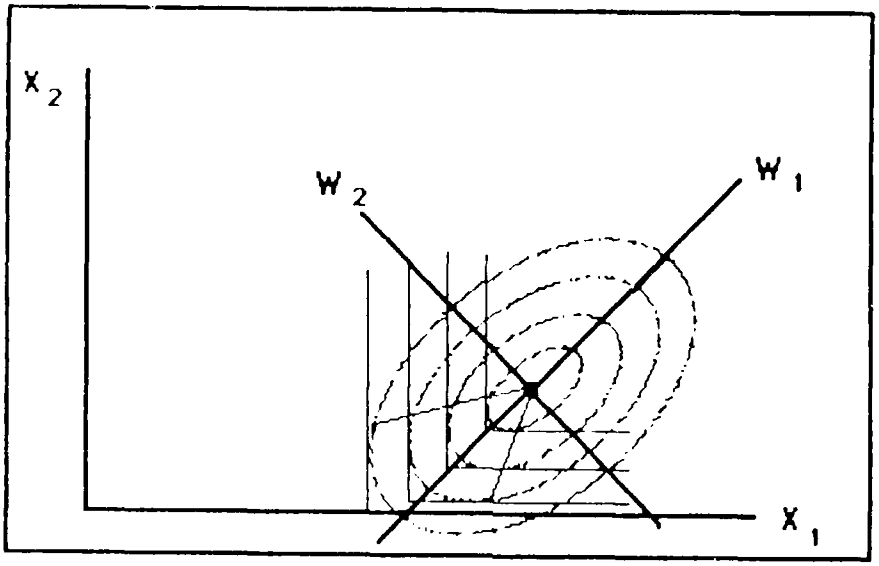

Fig 7 Isoquant Map of Sortie Production

This feasible region is where the isoquants are negatively sloped (diminishing rate of technical substitution) and concave toward the origin. This is the only region where production would be possible. Also, the values of both inputs must be positive. Now the different combinations of the two inputs along the individual isoquants can be observed which produce each constant level of output. The marginal rate of technical substitution (RTS) or the rate at which x_2 can be substituted for x_1 while holding output constant along an individual isoquant can be seen (30:244). This can be shown by the following:

$$RTS\ (x_2 \text{ for } x_1) = -\, dx_1/dx_2|_Q \qquad (102)$$

The value of this RTS depends on the point on the isoquant where the slope is being measured.

A difference between the second order polynomials and the Cobb-Douglas or the CES production functions can now be shown. Neither the Cobb Douglas nor the CES functions pass through a relative maximum. Therefore, there are no boundary limitations on their respective isoquants. A polynomial production function, however, does pass through a maximum point and so the isoquants are limited up to that maximum point. For each contour of constant production, the number of possible input combinations will decrease until there is one single combination which will produce a "true" maximum output. This maximum output may be obtained only if the budget constraint imposed on the system allows the process to be operated there.

Conclusions

Based on the results of this research effort, it was shown that response surface methodology techniques can be used in constructing a second-order polynomial as a production function. Sufficient conditions for output maximization were outlined to ensure that the polynomial behaves as a production function.

Secondly, using the same response surface techniques, a cost constraint could be imposed on the system. Combining a plot of the cost contours with the yield contours showed how trade-offs can be made in the input combinations to move to a region of lower costs. The exact path from the point of maximum yield to the point of minimum cost was constructed. The restrictions on this method were outlined and an alternative method was discussed.

Thirdly, only two factors (support kits and maintenance crews) were used as inputs to resemble capital (kits) and labor (crews) that are normally

analyzed as the two major factors of production in economics. Through this research, the greater flexibility of using a variable elasticity of substitution function like a second-order polynomial was shown. Comparing this flexibility to the constraints imposed by the Cobb-Douglas and CES production functions hopefully will encourage the use of response surface techniques to develop second-order polynomials which can be used to model production processes.

Finally, the problems of using computer simulations to model input-output processes were shown. If non-profit organizations, like the Air Force, wish to model themselves as producers, then they must understand the limitations to their model's production functions as outlined in this research.

Recommendations

There are three areas where further research may be suggested. The first is to find an Air Force simulation model that is currently used to allocate resources and see if, by using the techniques described in this research, that system adheres to modeling as a production function. It has been shown in this research that response surface techniques can easily be adopted to the estimation of polynomials as production functions. An interesting area of research would be to see if a current system is operating under the optimal conditions described in this research.

The simulation model used in this research was a simple example to show the methodology to be used if this technique were applied. This leads to the second area of further research. By using a current operational simulation model that might possibly involve more than two inputs, further research could be done by applying the output-maximization, cost-minimization analysis to three or more inputs. Developing the cost-minimization

constraints (first- and second-order conditions) for this more complex model would indeed be challenging. Also, it would be interesting to see how the response surfaces react to the interaction of more than two factors.

The final recommendation deals with graphing the actual elasticity of substitution curves for the VES functions developed through the response surface analysis. If "good" data were available from a system that was modeled using a Cobb-Douglas or a CES function, it would be interesting to compare the results of those production functions with production functions built by the response surface approach outlined in this research. Graphing the VES curves and comparing the input trade-offs seen on an elasticity plot with the trade offs shown on the response surface contours could be further proof of the benefit of using the techniques described in this research effort. Also, additional benefits of using second-order polynomials as production functions could be shown when an actual comparison is made to a system that is currently modeled using a Cobb-Douglas or a CES production function.

Appendix

Resource-Allocation Model

This appendix contains the SLAM II and FORTRAN code of the resource-allocation simulation model used in this research. The first section lists the SLAM II code and the second section lists the FORTRAN code.

SLAM Code

```
;   LTC EBELING'S CAPABILITY MODEL MODIFIED TO RUN
;   EXPERIMENTAL DESIGNS USING AC.FOR. MODIFIED
;   BY CAPT J. REVETTA.
;
GEN,EBELING,CAPABILITY MODEL,1/28/85,22,N,N,,N,N,72;

LIMITS,9,3,200;
TIMST,XX(1);
TIMST,NNQ(1);
TIMST,NNACT(4);
;
;   TIME UNIT IS ONE HOUR
NETWORK;
         RESOURCE/1,WRSK(0),1;    SET LEVEL OF SPARES
         RESOURCE/2,CREW(0),2,4;  ASSIGN MAINTENANCE CREWS
         RESOURCE/3,BOMBS(0),5;   DEFINE INITIAL BOMB LVL
         GATE/STORM,OPEN,6;       MODEL WEATHER
         GATE/DAY,CLOSED,7;       DAYLIGHT FLYING
;
;   MODEL SEGMENT I               ****SORTIE GENERATION****
;                                ********MAIN NETWORK*******
;
MSN      AWAIT(2),CREW;           WAIT FOR CREWCHIEF
         ACT/2,RLOGN(1,.5,4);     PREFLT
         FREE,CREW;               RELEASE CREWCHIEF
         AWAIT(7),DAY;            WAIT FOR DAYLIGHT
FLY      AWAIT(6),STORM;          WAIT FOR GOOD WEATHER
         AWAIT(5),BOMBS/6;        NEED MUNITIONS
         ALTER,BOMBS/-6;
         FREE,BOMBS/6;
         QUEUE(3);                WAIT FOR LOAD CREW
         ACT(4)/3,EXPON(.5);      BOMB LOAD
         GOON;
         ACT,,.10,FMAIN;          GROUND ABORT
         ACT,.2,.90;              LAUNCH AIRCRAFT
         GOON;
         COLCT(1),BETWEEN,MTB SORTIES,10/.5/.5;
         COLCT(2),INT(1),TURN TIME,20/1/1;
         ACT/4,RNORM(2,.5,.4);SORTIE    FLY MISSION (SORTIE)
```

```
        ASSIGN,ATRIB(1)=TNOW,
        GOON,
        ACT,,.01,ATRIT;            ATTRIT ACFT
        ACT,,.05,BATL;             BATTLE DAMAGE
        ACT,2,.94;                 TURN ACFT
        GOON;
        AWAIT(2),CREW;             AWAIT CREWCHIEF
        ACT/5,RLOGN(.5,.25,4);THRUFLT (POST FLT EXAMINATION)
        FREE,CREW;                 RELEASE CREWCHIEF
        GOON;
        ACT,,.30,MAINT;            UNSCHED MAINT
        ACT,,.70;
        GOON,1;
        ACT,,NNGAT(DAY).EQ.0,FLY,FLY AGAIN
        ACT,,,MSN;                 PREFLIGHT FOR AM MSN
;
; MODEL SEGMENT II          ****ACFT ATTRITION****
;
ATTRIT GOON;                       LOSE AN ACFT
        ACT/6,ATTRITS
        ASSIGN,XX(1)=XX(1)-1;
        TERM;
;
; MODEL SEGMENT III         ****UNSCHED MAINT****
;
FMAIN ALTER,BOMBS/6;               RETURN MUNITIONS
MAINT ASSIGN,XX(1)=XX(1)-1;
        AWAIT(4),CREW;             GET A CREWCHIEF
        ACT,RLOGN(1,.25),.70,RR,   TROUBLESHOOT
        ACT,RLOGN(1,.25),.30;
        GOON;
        ACT/7,RLOGN(2,.5,4);MINOR RPR     MINOR REPAIR
        FREE,CREW;
        ASSIGN,XX(1)=XX(1)+1;
        ACT,,MSN;                  GO TO PREFLIGHT
RR      AWAIT(1),WRSK;             WAIT FOR PART
        ACT/9,RLOGN(4,.5,4);R&R WRSK      REMOVE AND REPLACE
        FREE,CREW;
        ASSIGN,XX(1)=XX(1)+1;
        ACT,,,MSN;                 GO TO PREFLIGHT
        ACT;                       PART GOES TO SHOP
        QUEUE(8);                  ENTER SHOP REPAIR
        ACT(3)/8,EXPON(5),REPAIR PART- NOTE 3 CREWS
        FREE,WRSK;
        TERM;
;
; MODEL SEGMENT IV          ****WEATHER****
;
        CREATE,UNFRM(18,30),,,1;   A STORM EVERY 24 HRS (AVG)
CLS     CLOSE,STORM;
        ACT/1,UNFRM(1.5,2.5);STORM        STORM LASTS 2 HRS
        OPEN,STROM;
        ACT,UNFRM(18,30),,CLS;
;
```

```
; MODEL SEGMENT V            ****MUNITIONS****
;
       CREATE,UNFRM(18,30);      CONVOY ARRIVES DAILY
       ASSIGN,XX(3)-1;
BRN    GOON,
       ACT/11,RLOGN(2,1),BOMB BUILD      BUILDUP AND LINE DELIVERY
       ALTER,BOMBS/44                LOT SIZE
       ASSIGN,XX(3)-XX(3)+1;
       ACT,,XX(3).LT.6,BRN; NOTE 5 TRUCKS PER DAY UNLOADED
       ACT;
       TERM;
;
; MODEL SEGMENT VI           ****DAY/NIGHT SHIFT****
;
       CREATE,,12;
BACK   OPEN,DAY;
       ALTER,CREW/6;              DAY SHIFT COMES ON DUTY
       ACT,12;                    12 HRS OF DAYLIGHT
       CLOSE,DAY;                 NIGHT HAS FALLEN
       ALTER,CREW/-6;             NIGHTTIME WORK FORCE
       ACT,12,,BACK;
;
; MODEL SEGMENT VII          ****BATTLE DAMAGE NETWORK****
;
BATL   GOON;
        ASSIGN,XX(1)-XX(1)-1;
        ACT/10,UNFRM(22,26),CLSS RPR        CLSS REPAIR 24 HR AVG
        GOON;
        ASSIGN,XX(1)-XX(1)+1;
        ACT,,,MSN;
;
        ENDNETWORK;
INIT,0,720;         SIMULATE 30 DAYS
SIMULATE;
   24     8      9  288
SEEDS,+76071161363,+63709980095,+20315910979,
     +705098259815;
SIMULATE;
   24     8      9  288
SEEDS,+286360060485,+71077659415,+82692713921,
     +36687465289;
SIMULATE;
   24     8      9  288
SEEDS,+76071161363,+63709980095,+20315910979,
     +705098259815;
SIMULATE,
   24     8      9  288
SEEDS,+286360060485,+71077659415,+82692713921,
     +36687465289,
SIMULATE,
   24     8      9  288
SEEDS,+76071161363,+63709980095,+20315910979,
     +705098259815,
SIMULATE,
```

```
  24      8       9   288
SEEDS,+28636060485,+71077659415,+82692713921,
      +36687465289;
SIMULATE;
  24      8       6   288
SEEDS,+76071161363,+63709980095,+20315910979,
      +705098259815;
SIMULATE;
  24      8       6   288
SEEDS,+28636060485,+71077659415,+82692713921,
      +36687465289;
SIMULATE;
  24      8      12   288
SEEDS,+76071161363,+63709980095,+20315910979,
      +705098259815;
SIMULATE;
  24      8      12   288
SEEDS,+28636060485,+71077659415,+82692713921,
      +36687465289;
SIMULATE;
  24      6       9   288
SEEDS,+76071161363,+63709980095,+20315910979,
      +705098259815;
SIMULATE;
  24      6       9   288
SEEDS,+28636060485,+71077659415,+82692713921,
      +36687465289;
SIMULATE;
  24      6       6   288
SEEDS,+76071161363,+63709980095,+20315910979,
      +705098259815;
SIMULATE;
  24      6       6   288
SEEDS,+28636060485,+71077659415,+82692713921,
      +36687465289;
SIMULATE;
  24      6      12   288
SEEDS,+76071161363,+63709980095,+20315910979,
      +705098259815;
SIMULATE;
  24      6      12   288
SEEDS,+28636060485,+71077659415,+82692713921,
      +36687465289;
SIMULATE;
  24     10       6   288
SEEDS,+76071161363,+63709980095,+20315910979,
      +705098259815;
SIMULATE;
  24     10       6   288
SEEDS,+28636060485,+71077659415,+82692713921,
      +36687465289;
SIMULATE;
  24     10       9   288
SEEDS,+76071161363,+63709980095,+20315910979,
```

```
        +705098259815;
SIMULATE;
   24     10        9   288
SEEDS,+28636060485,+71077659415,+82692713921,
        +36687465289;
SIMULATE;
   24     10       12   288
SEEDS,+76071161363,+63709980095,+20315910979,
        +705098259815;
SIMULATE;
   24     10       12   288
SEEDS,+28636060485,+71077659415,+82692713921,
        +36687465289;
FIN;
```

FORTRAN Code

```
      PROGRAM MAIN
      DIMENSION NSET(10000)
      COMMON/SCOM1/ATRIB(100),DD(100),DDL(100),DTNOW,II,MFA,MSTOP,NCLNR
     1,NCRDR,NPRNT,NNRUN,NNSET,NTAPE,SS(100),SSL(100),TNEXT,TNOW,XX(100)
      COMMON/UCOM1/IAC,IWRSK,ICREW,IBOMBS
      COMMON QSET(10000)
      NNSET=10000
      NCRDR=5
      NPRNT=6
      NTAPE=7
      NPLOT=2
      OPEN(10,STATUS='MEW',FILE='CAP.DAT',FORM='FORMATTED')
      CALL SLAM
      STOP
      END
C
C
      SUBROUTINE INTLC
      COMMON/SCOM1/ATRIB(100),DD(100),DDL(100),DTNOW,II,MFA,MSTOP,NCLNR
     1,NCRDR,NPRNT,NNRUN,NNSET,NTAPE,SS(100),SSL(100),TNEXT,TNOW,XX(100)
      COMMON/UCOM1/IAC,IWRSK,ICREW,IBOMBS

C  READ IN STARTING VALUES FOR EXPERIMENT
      DIMENSION A(5)

      READ (NCRDR,100) IAC,IWRSK,ICREW,IBOMBS
100   FORMAT (4I5)

      XX(1)=IAC                ! SET AIRCRAFT LEVEL
      CALL ALTER(1,IWRSK)   ! SET LEVEL OF SPARES
      CALL ALTER(2,ICREW)   ! NUMBER OF MAINT ICREWS
      CALL ALTER (3,IBOMBS) !  INITIAL BOMB LEVEL

      DO 10 I=1,IAC
         A(1)=TNOW
         CALL FILEM(2,A)
```

```
10   CONTINUE

     PRINT *,IAC,IWRSK,ICREW,IBOMBS

     RETURN
     END
C
C
     SUBROUTINE OTPUT
     COMMON/SCOM1/ATRIB(100),DD(100),DTNOW,II,MFA,MSTOP,NCLNR
    1,NCRDR,NPRNT,NNRUN,NNSET,NTAPE,SS(100),SSL(100),TNEXT,TNOW,XX(100)
     COMMON/UCOM1/IAC,IWRSK,ICREW,IBOMBS

     WRITE(10,100) CNUM(1),IAC,IWRSK,ICREW,IBOMBS
100  FORMAT(' ',F8.2,1X,1X,4I5)
     RETURN
     END
```

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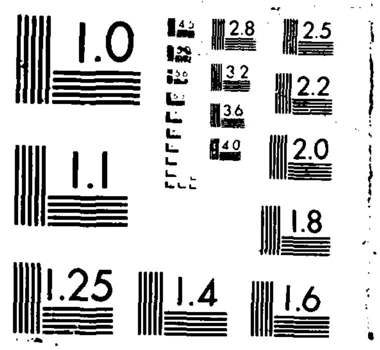
1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

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Associate Professor of Economics

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Box 19 (cont'd.)

This research attempted to show that an Air Force unit can be modeled as an industry with its "output" determined through a production function. A second-order polynomial was used as the production function in this research. A resource-allocation simulation was used to generate the data for analysis. Only two factors were analyzed--support kits and maintenance crews. In this way, these two inputs could be compared to the microeconomic factors of production--capital and labor.

Basic Response Surface Methodology (RSM) techniques were used to estimate the second-order polynomial. Experimental designs in the form of central composite designs(CCD) were used to determine the input factor combinations. A complete statistical analysis of the pure linear model and the second-order model, complete with statistical tests and ANOVA, was performed. Basic microeconomic definitions of first- and second-order conditions were discussed and the conditions for least-cost combinations of the inputs for the second-order polynomial were derived.

A canonical analysis was done on the output data in order to plot the response contours of maximum yield. Also, a cost constraint was imposed on this production function and a multiple response surface with maximum yield and minimum cost contours overlapped within the experimental region was plotted to show the relationship of maximum yield to minimum cost.

The results of the canonical analysis of the response model indicated that a production function can be maximized subject to a minimum cost constraint through the use of a multiple response system. The path from maximum yield to minimum cost, and the trade-offs involved, were discussed. Also, the problems associated with using simulated data to estimate production functions were outlined. Finally, some benefits of using a second-order polynomial as a production function in comparison with the commonly used Cobb-Douglas and Constant Elasticity of Substitution (CES) production function were discussed.

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